Chapter 4

Counting Rules and Probability

1 Counting Rules

Many times one wishes to know the number of all possible outcomes for a sequence of events. To determine this number, three rules can be used: the *fundamental counting rule, the permutation rule,* and the *combination rule.*

1.1 Fundamental Counting Rule

Suppose k events $E_1, E_2, \ldots, E_k$ are to be performed in such a way that each event is independent of the others. If event E_1 can be performed in $n(E_1)$ ways, event E_2 in $n(E_2)$ ways, and so on, then the total number of ways of performing all k events is given by the product

$$N = n(E_1)n(E_2)\cdots n(E_k)$$

Example 1

A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

Solution:

Since the coin can land either heads up or tails up and since the die can land with any one of six numbers showing face up, there are $2 \cdot 6 = 12$.

Example 2

A paint manufacturer wishes to manufacture several different paints. The categories include

Color	Red, blue, white, black, green, brown, yellow
Type	Latex, oil
Texture	Flat, semigloss, high loss
Use	outdoor, indoor

How many different kinds of paint can be made if a person can select one color, one type, one texture, and one use?

Solution:

Color		Type		Texture		Use		
7	×	2	×	3	×	2	=	**84**

Example 3

There are four blood types, A, B, AB, and O. Blood can also be Rh+ and Rh-. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labeled?

Solution:

Since there are 4 possibilities for blood type, 2 possibilities for Rh factor, and 2 possibilities for the gender of the donor, there are $4 \cdot 2 \cdot 2$ or 16, different classification categories, as shown.

Blood type		Rh		Gender	
4	×	2	×	2	= **16**

Example 4

The digits 0, 1, 2, 3, and 4 are to be used in 4 digits ID card. How many different cards are possible if repetitions are permitted?

Solution:
Since there are 4 spaces to fill and 5 choices for each space, the solution is

$$5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$$

Remark From this example, if repetitions are not permitted, the first digit can be chosen in five different ways, the second in 4 different ways, etc. Thus the solution is $5 \cdot 4 \cdot 3 \cdot 2 = 120$.

Example 5

A 'word' is an ordered list of letters.

a. How many 2-letter words can be made using the letters of the alphabet?

b. How many 3-letter words can be made in which the second letter is a is wowel?

Solution

Let $A = \{x \mid x \text{ is a letter of the alphabet}\}$. Then the number of elements in A is $n(A) = 26$.

a. A 2-letter word can be thought of as the sequence of events *AA*. Thus, the number of possible 2-letter words is $n(A)n(A) = 26^2$.

b. If $B = \{\text{a, e, i, o, u}\}$ be the set of vowels, then $n(B) = 5$. The 3-letter words in which the second letter is a vowel correspond to the sequence of events *A B A*. The number of such words in $n(A)n(B)n(A) = 26 \times 5 \times 26 = 3380$.

Two other rules that can be use to determine the total number of possibilities of a sequence of events is the permutation rule and the combination rule. These rules use *factorial notation.*

1.2 Permutations

A **permutation** is an arrangement of objects in a specific order.

Permutation Rule

The permutation of *n* objects taken *k* at a time is an ordered arrangement of *k* of the *n* objects selected **without repetition.** The total number of these arrangements is denoted by $P(n,k)$. The formula is

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Example 7

Compute $P(6,4)$, $P(5,5)$

Solution

$$P(6,4)=\frac{6!}{(6-4)!}=\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1}=6\cdot5\cdot4\cdot3=360$$

$$P(5,5)=\frac{5!}{(5-5)!}=\frac{5!}{0!}=\frac{5\cdot4\cdot3\cdot2\cdot1}{1}=120$$

Example 8

A television news director wishes to use three news stories on an evening show. One story will be the lead story, one will be the second story, and the last will be a closing story. If the director has a total of eight stories to choose from, how many possible ways can the program be set up?

Solution

$$P(8,3)=\frac{8!}{(8-3)!}=\frac{8!}{5!}=336 \text{ ways}$$

Example 9

How many different ways can a chairperson and an assistant chairperson be selected for a research project if there are seven scientist available?

Solution:

$$P(7,2)=\frac{7!}{(7-2)!}=\frac{7!}{5!}=42$$

Permutation with Repetition

When the object can be selected repeatedly, the total number of permutations is determined by

$$P(n,k)=\underbrace{n\cdot n\cdot\cdots\cdot n}_{\text{k terms}}=n^k$$

Example 10

How many 5-digit numbers can be made with the digits 1 through 8 if:

a. No digit is repeated? **b.** Repeated digits are allowed?

Solution

a. $P(8,5)=\frac{8!}{(8-5)!}=6720$

b. $8^5=32768$

Example 11

From a set consisting of 10 digits, 26 lower-case and 26 capital letters, one can create $P(62,8)=136,325,893,334,400$ different 8-character passwords.

1.3Combinations

A **combination** of *n* objects taken *k* at a time is a subset of *k* objects selected without regard to order from a collection of *n* objects. The total number of such combination is denoted by $C(n,k)$, ${}_nC_k$, or $\binom{n}{k}$ and read as *n* choose *k*.

$$ {}_nC_k = \frac{n!}{k!(n-k)!} $$

Example 11

How many combinations of 4 objects are there, taken 2 at a time?

Solution

$$ C(4,2) = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = 6 $$

Example 12

A bicycle shop owner has 12 mountain bicycles in the showroom. The owner wishes to select 5 of them to display at a bicycle show. How many different ways can a group of 5 be selected?

Solution

$$ C(12,5) = \frac{12!}{(12-5)!5!} = \frac{12!}{7!5!} = 792 $$

Example 13

In a club there are 7 women and 5 men, A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

Solution

$$ C(7,3) \times C(3,2) = \frac{7!}{(7-3)!3!} \times \frac{3!}{(3-2)!2!} = 350 $$

Example 14

An interior decorator has 8 shades of paint from which to choose 2 shades, 5 rugs from which to choose 3, and 7 light fixtures from which to choose 4. How many different combinations are there altogether?

Solution

There are $C(8,2) \times C(5,3) \times C(7,4) = 9800$ combinations together.

Example 14

An antivirus software reports that 3 folders out of 10 are infected. How many possibilities are there?

Solution

$$ \binom{10}{3} = \frac{10!}{3!(10-3)!} = 120 $$

2 Basic Concept of Probability

2.1 Experiment, Outcomes, Events and Sample Space

An **experiment**, also called a **random experiment**, is a activity in which the result cannot be predicted with certainty, and each repetition of an experiment is called a **trial**.

An **outcome** is a result of an experiment. An **event** is any collection of outcomes. A **simple event** is an event with only one possible outcome. A **compound event** consists of two or more outcomes or simple events. The **sample space** for a given experiment is a set *S* that contains all possible outcomes of the experiment.

When a coin is tossed, there are two possible outcomes: head (H) or tail (T). In the roll of a single die, there are six possible outcomes: 1, 2, 3, 4, 5, or 6. In any experiment, the set of all possible outcomes is called the *sample space.*
Some sample spaces for various probability experiments are shown here:

Experiment	**Sample Space**
Toss one coin	H, T
Roll a die	1, 2, 3, 4, 5, 6
Answer a True/false question	True, False
Toss two coins	HH, TT, HT, TH

Example 1

Find the sample space for rolling two dice.

Solution

Each die can land in six different ways, and two dice, and two dice are rolled then the sample space is the list of pairs of numbers below

Die 1 / Die 2	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6,6)

Example 2

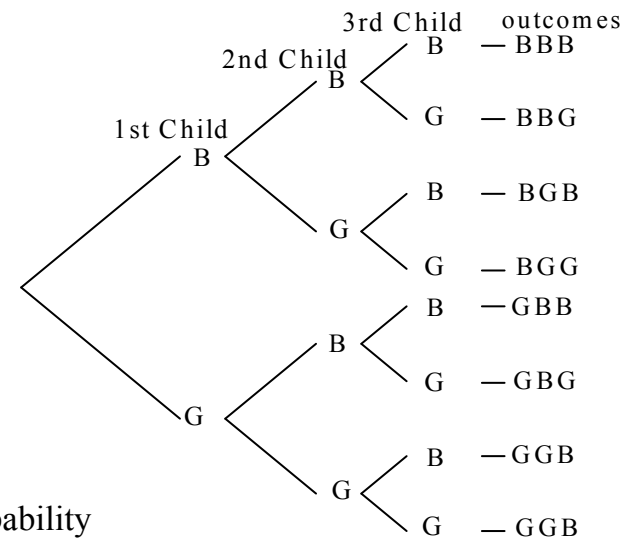
Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl

Solution

The possibilities are

BBB, BBG, BGB, GBB, GGG, GGB, GBG, BGG

For this example, there is another way to find all possible outcomes of a probability experiment. It is a tree diagram.



2.2 Probability

Probability measures the likelihood of the occurrence of an event. The probability of an event can be thought of as the **relative frequency** of the event; that is, the fraction of the time that the event can be expected to occur if the experiment is repeated over and over many times. Probabilities are **number between 0 and 1**. An event with probability 0 has **no chance of occurring,** and an event with probability 1 **is certain to occur.** In particular, the sample space has probability 1.

For experiments such as tossing coins, rolling dice, and random sampling, in which the outcomes are equally likely to occur, there is a simple formula for obtaining probability, which we call the **basic probability** (also classical probability) formula.

In an experiment for which the sample space is *S* and the outcomes are equally likely, the probability of an event *E* occurring is given by the formula

$$P(E) = \frac{n(E)}{n(S)}$$

In particular, $P(S) = 1$.

Example 3

For a card drawn from an ordinary deck, find the probability of getting a queen.

Solution

Since there are 52 cards in a desk and there are 4 queens, then

$$P(\text{queen}) = \frac{4}{52} = \frac{1}{13}$$

Example 4

If a family has three children, find the probability that all the children are girls.

Solution

The sample space for the gender of children for a family that has three children is BBB, BBG, BGB, GBB, GGG, GGB, GBG, and BGG. Since there is one way in eight possibilities for all three to be girls,

$$P(\text{GGG}) = \frac{1}{8}$$

Example 5

A card is drawn from an ordinary deck. Find these probabilities

a. Of getting a jack
b. Of getting the 6 of clubs
c. Of getting a 3 or a diamond.

Solution

a. $P(\text{jack}) = \frac{4}{52} = \frac{1}{13}$

b. $P(6 \text{ of clubs}) = \frac{1}{52}$

c. There are four 3s and 13 diamonds, but the 3 of diamonds is counted twice in the listing. Hence, there are 16 possibilities of drawing a 3 or a diamond, so

$$P(3 \text{ or diamond}) = \frac{16}{52} = \frac{4}{13}$$

Example 6

In a soft drink survey, the researcher asked 25 people if they like the taste of the new soft drink. The responses were classified as “yes,” “no,” or “undecided.”

Response	Frequency
Yes	15
No	8
Undecided	2
	Total 25

Find the probability that a person responded no.

Solution

$$P(E)=\frac{f}{n}=\frac{8}{25}$$

Example 7

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities:

a. A person has type O blood
b. A person has type A or type B blood
c. A person has neither type A nor type O blood

Type	Frequency
A	22
B	5
AB	2
O	21
	Total 50

a. $P(\text{O})=\frac{f}{n}=\frac{21}{50}$

b. $P(\text{A or B})=\frac{22}{50}+\frac{5}{50}=\frac{27}{50}$ (Add the frequencies of the two classes)

c. $P(\text{neither A nor O})=\frac{5}{50}+\frac{2}{50}=\frac{7}{50}$

(Neither A nor O means that a person has either type B or type AB blood)

3 Laws of Probability

Two events A and B are said to be **mutually exclusive** if they have no outcomes in common; that is $A\cap B$ cannot occur.

The Probability of a Union

If $A=A_1\cup A_2\cup\ldots\cup A_k$ where $A_1,A_2,\ldots,A_k$ are mutually exclusive events, then

$$P(A)=P(A_1)+P(A_2)+\cdots+P(A_k)$$

In particular, is A=S, $P(A_1)+P(A_2)+\cdots+P(A_k)=1$

If A and B are any two events, then

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

More generally, if A_1, A_2, and A_3 are any three events, then

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

Example 1

Determine which events are mutually exclusive and which are not when a single die is rolled.

a. Getting an odd number and getting an even number
b. Getting a 3 and getting an odd number
c. Getting an odd number and getting a number less than 4
d. getting a number greater than 4 and getting a number less than 4

Solution

a. and d. are mutually exclusive. b. and d. are not.

Example 2

A day of a week is selected at random. Find the probability that it is weekend day.

Solution

$$P(\text{Saturday or Sunday}) = P(\text{Saturday}) + P(\text{Sunday}) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

Example 3

A single card is draw from a deck. Find the probability that it is a king or a club.

Solution

$$P(\text{king or club}) = P(\text{king}) + P(\text{club}) - P(\text{king of clubs})$$
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$$

Example 4

In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

Solution

The sample space is shown here

Staff	Females	Males	Total
Nurses	7	1	8
Physicians	3	2	5
Total	10	3	13

The probability is

$$P(\text{nurse or male}) = P(\text{nurse}) + P(\text{male}) - P(\text{male nurse})$$
$$= \frac{8}{13} + \frac{3}{13} - \frac{1}{13} = \frac{10}{13}$$

The Probability of a Complement

For any event A, $P(A) = 1 - P(A^c)$

Example 5

If 5 cards are dealt at random from a standard deck of 52 cards, what is the probability that at least one is an ace?

Solution

Let A be the event of having no ace, then the probability we seek is $P(A^c)$.

Number of way of drawing a hand having no ace is $C(48,5)$, hence

$$P(A)=\frac{C(48,5)}{C(52,5)}=0.659$$

and probability that there is at least one ace is

$$P(A^c)=1-P(A)=1-0.659=0.341$$

Example 6

Suppose you buy 10 tickets to a lottery in which there are to be 5 prizes awarded on the basis of random drawing. Suppose that there are a total of 150 tickets sold. What are the chances of winning?

a.No prices? **b.** At least one prize? **c.** Exactly 3 prizes? **d.** All 5 prizes?

Solution

a. $P(\text{No prizes})=\dfrac{C(140,5)}{C(150,5)}=0.705$

b. $P(\text{at least 1 prize})=1-P(\text{No prize})=1-0.705=0.295$

C. $P(\text{exactly 3 prizes})=\dfrac{C(10,3)\times C(140,2)}{C(150,5)}=0.002$

d. $P(\text{all 5 prizes})=\dfrac{C(10,5)}{C(150,5)}=4\times10^{-7}$

4 Conditional Probability and Bayes'Rule

Sometimes the occurrence of one event affects the probability of another event, and the probability whose calculation is based on the assumption that a particular event occurs is called a **conditional probability.**

The Conditional Probability Formula

For any events *A* and *B*, with $P(B)\neq 0$,

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}$$

whence, we can obtain the formula

$$P(A\cap B)=P(A\mid B)\times P(B)$$

which is known as product rule.

Independent Events

If $P(B\mid A)=P(B)$; that is, the probability of *B* occurring is not affected by the occurrence or non-occurrence of *A*, then we say that *A* and *B* are independent events. This is equivalent to

$$P(A\cap B)=P(A)\times P(B)$$

Example 7

A box contains 3 green and 2 yellow marbles. Two marbles are selected. Find the probability that both selected marbles are green in the case of: **a.** No replacement (the first marble is not replaced before the second is chosen.) **b.** Replacement (the first marble is replaced before the second is chosen.)

Solution

A={the first marble is green} *B*={the second marble is green}

a. $P(A\cap B)=P(A)\times P(B\mid A)=\dfrac{3}{5}\times\dfrac{2}{4}=\dfrac{3}{10}$

b. $P(A \cap B) = P(A) \times P(B) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$

Example 8

In a certain state, 55% vote for Spiff for governor. Three votes are selected at random. **a.** Find the probability that all 3 will vote for Spiff. **b.** Find the probability that exactly 2 will vote for Spiff.

Solution

Let *S* denote the event that a selected voter will vote for Spiff and let *N* denote the event that the voter does not vote for Spiff. On any selection, $P(S) = 0.55$, and $P(N) = 0.45$.

a. $P(SSS) = P(S \cap S \cap) = P(S) \times P(S) \times P(S) = 0.55^3 = 0.166$

b. We have three ways: *S S N*, *S N S*, or *N S S*. Since

$P(S\,S\,N) = P(S\,N\,S) = P(N\,S\,S) = P(S) \times P(S) \times P(N) = 0.55^2 \times 0.45$

Hence,

$P(\text{exactly 2...}) = P(SSN) + P(SNS) + P(NSS) = 3 \times 0.55^2 \times 0.45 = 0.408$

Example 9

A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip and a white chip is $\frac{15}{56}$, and the probability of selecting a black chip on the first draw is $\frac{3}{8}$, find the probability of selecting white chip on the second draw, given that the first chip selected was a black chip.

Solution:

Let *B* be the event of selecting a black chip, and *W* the event of selecting white chip

Then

$$P(W \mid B) = \frac{P(B \text{ and } W)}{P(B)} = \frac{\frac{15}{56}}{\frac{3}{8}} = \frac{5}{7}$$

Example 10

A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	**Yes**	**No**	**Total**
Male	32	18	50
Female	8	42	50
Total	40	60	100

Find these probabilities

a. The respondent answered yes, given that the respondent was a female
b. The respondent was a male, given that the respondent answered no.

Solution

Let *M*: respondent was a male *Y*: respondent answered yes
F: respondent was a female *N*: respondent answered no

a. The problem is to find $P(Y \mid F)$. The rule state

$$P(Y \mid F) = \frac{P(F \text{ and } Y)}{P(F)} = \frac{\frac{8}{100}}{\frac{50}{100}} = \frac{4}{25}$$

b. The problem is to find $P(M \mid N) = \dfrac{P(N \text{ and } M)}{P(N)} = \dfrac{\frac{18}{100}}{\frac{60}{100}} = \dfrac{3}{10}$

Example 11

A survey of attitudes about one's job yields the data tin the following table

	Happy	Unhappy	Total
Bus drivers	50	75	125
Lawyers	40	35	75
Total	90	110	200

A person from this group is selected at random. Given that the selected person is a bus driver, find the probability that he or she is happy. (Answer: 0.4)

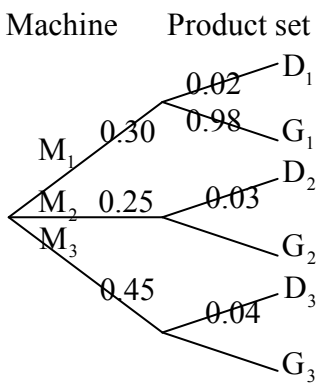
Bayes' Rule[1]

Let $A_1, A_2, \ldots, A_n$ be mutually exclusive events whose union is the sample space *S*, and let *E* be an event with $P(E) \neq 0$. Then

$$P(A_j \mid E) = \frac{P(A_j) \times P(E \mid A_j)}{P(A_1) \times P(E \mid A_1) + \cdots + P(A_n) \times P(E \mid A_n)}$$

Example 12

Machine M_1 produces 30% of the day's production of a certain product, while machine M_2 and M_3 produce 25% and 45%, respectively. Suppose machine M_1, M_2, and M_3 turn out 2%, 3% and 4% defective products, respectively, each day. If an item is picked at random and is found to be defective, what is the probability that it came from machine M_1?



$$P(\mathrm{M}_1 \mid \text{Defective}) = \frac{0.30 \times 0.02}{0.30 \times 0.02 + 0.25 \times 0.03 + 0.45 \times 0.04} = 0.19$$

Example 13

Suppose that 60% of a company's computer chips are manufactured in one factory (denoted *A*) and 40% are produced in other factory ($\overline{A}$). Suppose we now learn that the randomly selected chip is defective and the defect rates for the two factories are 35% (for A) and 25% (for $\overline{A}$) respectively. Given the defective chip, what is the probability it came from factory *A*? (Answer: 0.677)

[1] This rule is named for an English mathematician Thomas Bayes (1702-1761)

Example 14

On a game show, a contestant can select one of four boxes. Box 1 contains one $100 bills and nine $1 bills. Box 2 contains two $100 bills and eight $1 bills. Box 3 contains three $100 bills and seven $1 bills. Box 4 contains five $100 bills and five $1 bills. The contestant selects a box at random and selects a bill from the box at random. If a $100 bill is selected, find the probability that it came from box 4. (Answer: 0.455)