Lecture Note

Mathematics for Engineering III

Norton University
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Mathematics for Engineering III

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Chapter 1

Partial Differentiation

1 Functions of Two or More Variables

Definition

A **function of two real variables,** x and y is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D of the *xy*-plane.

A **function of three variables,** $x, y,$ and z is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D of three-dimensional space.

The set D in these definitions is the **domain** of the function; it is the set of points at which the function is defined.

In general, a function of n real variables, $x_1, x_2, \dots, x_n$, is regarded as a rule that assigns a unique real number $f(x_1, x_2, \dots, x_n)$ to each point $(x_1, x_2, \dots, x_n)$ in some set of n-dimensional space.

Example 1

$f:(x, y) \mapsto f(x, y) = 2x^2 y$ is a function of 2 variables. If x=1 and y=3, then the value of the function is $f(1,3) = 2\cdot 1^2 \cdot 3 = 6$.

Note We can denote $z = f(x_1, x_2, ..., x_n)$ and we call z the *dependent variable* and $x_1, x_2, ..., x_n$ *the independent variables.*

For the function of two variables $z = f(x, y)$, its domain is a set of point (x, y) of the *xy*-plane, on which $f(x, y)$ is defined. The set of point $P(x, y, z = f(x, y))$ represents the graph of $z = f(x, y)$. It is a surface in 3-space.

Example 2

State the domain of $z = f(x, y) = \sqrt{1 - x^2 - y^2}$

Solution

f is defined if $1 - x^2 - y^2 \geq 0 \Leftrightarrow x^2 + y^2 \leq 1$. Hence the domain of f is the points on the disc with radius of unity. (Fig. 1)

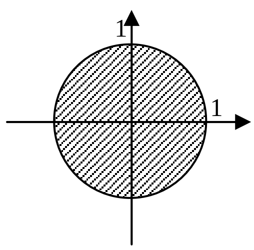


Fig.1

Example 3

Find the domain of $z = f(x, y) = \dfrac{1}{\sqrt{1 - x^2} \cdot \sqrt{1 - y^2}}$

Solution

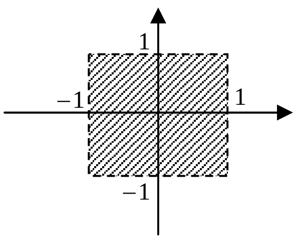
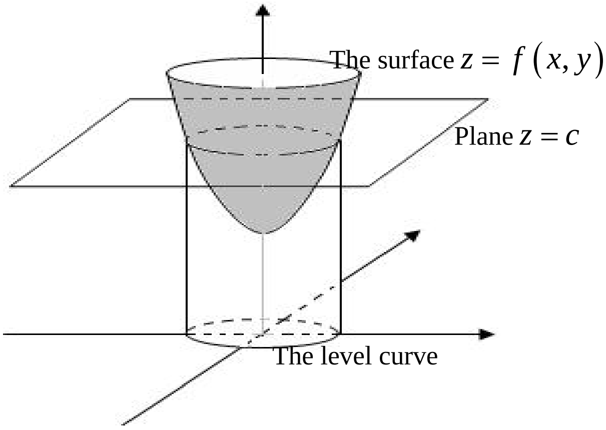


Fig. 2

z is defined if $\begin{cases}1-x^2>0\\1-y^2>0\end{cases} \Rightarrow \begin{cases}|x|<1\\|y|<1\end{cases}$. Hence the domain is the set of points inside of the rectangle. (See fig. 2)

Level Curves

Each horizontal plane $z = C$ intersects the surface $z = f(x,y)$ in a curve. The projection of this curve on xy-plane is called **a level curve**.



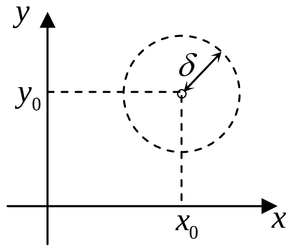
2 Limits and Continuity

Limit of a Function of Two Variables

The limit statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

means that for each given number $\varepsilon > 0$, there exists a number $\delta > 0$ so that whenever (x,y) is a point in the domain D of f such that



$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

then

$$|f(x,y) - L| < \varepsilon$$

or

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0 : \forall(x,y)\in D, 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

$$\Rightarrow |f(x,y) - L| < \varepsilon$$

N.b: If the $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ is not the same for all approaches or paths within the domain of f then the limit does not exist.

Example 1

Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2 + x - xy - y}{x - y}$

Solution

$$\text{For } x \neq y \;\; f(x,y) = \frac{x^2 + x - xy - y}{x - y} = \frac{(x+y)(x-y)}{x-y} = x + 1$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x+1) = 1$$

Example 2

If $f(x,y) = \dfrac{2xy}{x^2+y^2}$, show that $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist by evaluating this limit along the *x*-axis, the *y*-axis, and along the line $y = x$.

Solution

First note that the denominator is zero at $(0,0)$, so $f(0,0)$ is not defined. If we approach the origin along the *x*-axis (where $y = 0$), we find that

$$f(x,0) = \frac{2x(0)}{x^2+0} = 0$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = 0$ (and $x \neq 0$). If we approach the origin along the *y*-axis (where $x = 0$), we find that

$$f(x,0) = \frac{2(0)y}{0+y^2} = 0$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $x = 0$ (and $y \neq 0$).

However, along the line $y = x$, the functional values are

$$f(x,y) = f(x,x) = \frac{2x^2}{x^2+x^2} = 1 \text{ for } x \neq 0$$

so $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along $y = x$. Because $f(x,y)$ tends toward different numbers as $(x,y) \to (0,0)$ along the different paths, it follows that *f* has no limit at the origin $(0,0)$.

Example 3

Assuming each limit exists, evaluate:

a. $\lim_{(x,y)\to(3,-4)} \left(x^2+xy+y^2\right)$ (**ans:** 13) **b.** $\lim_{(x,y)\to(1,2)} \dfrac{2xy}{x^2+y^2}$ (**ans:** $\dfrac{4}{5}$)

Continuity of a Function of Two Variables

The function $f(x,y)$ is continuous at the point (x_0, y_0) if

(i). $f(x_0, y_0)$ is defined.

(ii). $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

(iii). $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

The function *f* is **continuous on a set** *S* if it is continuous at each point in *S*.

Limit and Continuity for function of three variables

The limit statement

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

means that for each $\varepsilon > 0, \exists \delta > 0$ such that $\left| f(x,y,z) - L \right| < \varepsilon$ whenever $f(x,y,z)$ is a point in the domain of f such that

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta$$

The function $f(x,y,z)$ is continuous at the point $P_0(x_0, y_0, z_0)$ if

(i). $f(x_0, y_0, z_0)$

(ii). $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$

(iii). $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0)$

3 Partial Derivatives

If $z = f(x,y)$ then the partial derivatives of *f* with respect to *x* and *y* are the function f_x and f_y, respectively defined by

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

provided the limits exist.

Example 1

$f(x,y) = x^3 y + x^2 y^2$. Find f_x, f_y

Solution

$$f_x(x,y) = 3x^2 y + 2xy^2$$

$$f_y(x,y) = x^3 + 2x^2 y$$

Alternative Notations for Partial Derivatives

For $z = f(x,y)$, the partial derivatives f_x, f_y are denoted by

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = z_x = D_x(f)$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x,y) = z_y = D_y(f)$$

The values of the partial derivatives of $f(x,y)$ at the point (a,b) are denoted by $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b)$ and $\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b)$

Example 2

Let $z = x^2 \sin(3x + y^3)$. Evaluate $\left.\frac{\partial z}{\partial x}\right|_{(\pi/3, 0)}$

Solution

$$\frac{\partial z}{\partial x} = 2x\sin\left(3x + y^3\right) + x^2\cos\left(3x + y^3\right)(3)$$
$$= 2x\sin\left(3x + y^3\right) + 3x^2\cos\left(3x + y^3\right)$$

Thus,

$$\left.\frac{\partial z}{\partial x}\right|_{(\pi/3,0)} = 2\left(\frac{\pi}{3}\right)\sin\pi + 3\left(\frac{\pi}{3}\right)^2\cos\pi$$
$$= \frac{2\pi}{3}(0) + \frac{\pi^2}{3}(-1) = -\frac{\pi^2}{3}$$

Example 3

Let $f(x, y, z) = x^2 + 2xy^2 + yz^3$. Determine: f_x, f_y and f_z.

Solution

We treat *y*, *z* as constants, then $f_x(x, y, z) = 2x + 2y^2$

We treat *x*, *z* as constants, then $f_y(x, y, z) = 4xy + z^3$

We treat *x*, *y* as constants, then $f_z(x, y, z) = 3yz^2$

Example 3

Let *z* be defined implicitly as a function of *x* and *y* by the equation $x^2z + yz^3 = x$
Determine $\partial z/\partial x$ and $\partial z/\partial y$

Solution

Differentiate implicitly with respect to *x*, treating *y* as a constant:

$$2xz + x^2\frac{\partial z}{\partial x} + 3yz^2\frac{\partial z}{\partial x} = 1$$

Then solve this equation for $\partial z/\partial x$:

$$\frac{\partial z}{\partial x} = \frac{1 - 2xz}{x^2 + 3yz^2}$$

Similarly, holding *x* constant and differentiating implicitly with respect to *y*, we find

$$x^2\frac{\partial z}{\partial y} + z^3 + 3yz^2\frac{\partial z}{\partial y} = 0$$

So that

$$\frac{\partial z}{\partial y} = \frac{-z^3}{x^2 + 3yz^2}$$

Partial Derivative as a slope

The line parallel to the *xz*-plane and tangent to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ has slope $f_x(x_0, y_0)$. Likewise, the tangent line to the surface at P_0 that parallel to the *yz*-plane has slope $f_y(x_0, y_0)$.

Example 4

Find the slope of the line that is parallel to the *xz*-plane and tangent to the surface $z = x\sqrt{x + y}$ at the point $P(1, 3, 2)$

Solution

If $f(x,y) = x\sqrt{x+y} = x(x+y)^{1/2}$ then the required slope is $f_x(1,3)$

$f_x(x,y) = \dfrac{x}{2\sqrt{x+y}} + \sqrt{x+y}$. Thus, $f_x(1,3) = \dfrac{9}{4}$

Higher-Order Partial Derivatives

Given $z = f(x,y)$, then

Second –order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

Mixed second-order partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

Note The notation f_{xy} means that we differentiate first with respect to *x* and then with respect to *y*, while $\dfrac{\partial^2 f}{\partial x \partial y}$ means just the opposite (differentiate with respect to *y* first and then
with respect to *x*).

Example 5

For $z = f(x,y) = 5x^2 - 2xy + 3y^3$, determine these higher-order partial derivatives.

a. $\dfrac{\partial^2 f}{\partial x \partial y}$ **b.** $\dfrac{\partial^2 f}{\partial y \partial x}$ **c.** $\dfrac{\partial^2 z}{\partial x^2}$ **d.** $f_{xy}(3,2)$

Solution

a. First differentiate with respect to *y*, then to *x*

$$\frac{\partial f}{\partial y} = -2x + 9y^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2x + 9y^2\right) = -2$$

b. Differentiate first with respect to *x* and then with respect to *y*.

$$\frac{\partial f}{\partial x} = 10x - 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(10x - 2y) = -2$$

c. Differentiate with respect to *x* twice:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(10x - 2y) = 10$$

d. Evaluate the mixed partial found in part **b** at the point $(3,2)$

$f_{xy}(3,2) = -2$.

Remark If the function $f(x,y)$ has mixed second-order partial derivatives f_{xy} and f_{yx} that are continuous in an **open disk** containing (x_0, y_0), then

$$f_{yx}(x_0, y_0) = f_{xy}(x_0, y_0)$$

In fact this remark is a theorem with the proof omitted here.

Example 6

Determine $f_{xy}, f_{yx}, f_{xx}, f_{xxy}$ where $f(x,y) = x^2ye^y$

Solution

We have the partial derivatives

$$f_x = 2xye^y \qquad f_y = x^2e^y + x^2ye^y$$

The mixed partial derivatives are

$$f_{xy} = (f_x)_y = 2xe^y + 2xye^y \qquad f_{yx} = x^2e^y + x^2ye^y$$

$$f_{xx} = (f_x)_x = 2ye^y \quad \text{and} \quad f_{xxy} = (f_{xx})_y = 2e^y + 2ye^y$$

Example 7

By direct calculation, show that $f_{xyz} = f_{yzx} = f_{zyx}$ for the function $f(x,y,z) = xyz + x^2y^2z^4$.

4 Directional Derivatives and Gradients

4.1 Directional Derivatives and Gradients of Two-Variable Function

Directional Derivative

Let *f* be a function of two variables, and let $\vec{u} = u_1\vec{i} + u_2\vec{j}$ be a unit vector. The **directional derivative of** *f* at $P_0(x_0, y_0)$ in the direction of $\vec{u}$ is given by

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided the limit exists.

Let $f(x,y)$ be a function that is differentiable at $P_0(x_0, y_0)$. Then f has a directional derivative in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ given by

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

Example 1

Find the derivative of $f(x,y) = 3 - 2x^2 + y^3$ at the point $P(1,2)$ in the direction of the unit vector $\vec{u} = \frac{1}{2}\vec{i} - \frac{\sqrt{3}}{2}\vec{j}$

Solution

The partial derivative $f_x(x,y) = -4x$ and $f_y(x,y) = 3y^2$. Then since $u_1 = \frac{1}{2}$ and $u_2 = -\frac{\sqrt{3}}{2}$, we have

$$D_{\vec{u}} f(1,2) = f_x(1,2)\left(\frac{1}{2}\right) + f_y(1,2)\left(-\frac{\sqrt{3}}{2}\right)$$
$$= -2 - 6\sqrt{3}$$

The Gradient

Let f be a differential function at (x, y) and let $f(x, y)$ have partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Then the **gradient** of f, denoted by ∇f, is a vector given by

$$\nabla f(x, y) = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$$

The value of the gradient at the point $P_0(x_0, y_0)$ is denoted by

$$\nabla f_0 = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$$

Example 2

Find the gradient of the function $f(x, y) = x^2 y + y^3$

Solution

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 y + y^3\right) = 2xy \qquad f_y(x, y) = \frac{\partial}{\partial y}\left(x^2 y + y^3\right) = x^2 + 3y^2$$

then

$$\nabla f(x, y) = 2xy\vec{i} + \left(x^2 + 3y^2\right)\vec{j}$$

Theorem:

If f is a differentiable function of *x* and *y*, then the directional derivative of f at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = \nabla f_0 \cdot \vec{u}$$

(The proof is consider an exercise)

Example 3

Find the directional derivative $f(x, y) = \ln\left(x^2 + y^3\right)$ at the point $P_0(1, -3)$ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j}$.

Solution

$$f_x(x, y) = \frac{2x}{x^2 + y^3}, \text{ so } f_x(1, -3) = -\frac{2}{26}$$
$$f_y(x, y) = \frac{3y^2}{x^2 + y^3}, \text{ so } f_y(1, -3) = -\frac{27}{26}$$

Thus, $\nabla f_0 = \nabla f(1, -3) = -\frac{2}{26}\vec{i} - \frac{27}{26}\vec{j}$

A unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} - 3\vec{j}}{\sqrt{2^2 + (-3)^2}} = \frac{1}{\sqrt{13}}\left(2\vec{i} - 3\vec{j}\right)$$

Thus

$$D_{\vec{u}}(x, y) = \nabla f_0 \cdot \vec{u} = \left(-\frac{2}{26}\right)\left(\frac{2}{\sqrt{13}}\right) + \left(-\frac{27}{26}\right)\left(-\frac{3}{\sqrt{13}}\right)$$

4.2 Directional Derivatives and Gradients of Three-Variable Function

Directional Derivatives

Let $f(x,y,z)$ be a differentiable function at the point $P_0(x_0,y_0,z_0)$, and let $\vec{u}=(u_1,u_2,u_3)$ be a unit vector. The directional derivative of *f* at the point P_0 in the direction of $\vec{u}$ is given by

$$D_{\vec{u}} f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)u_1 + f_y(x_0,y_0,z_0)u_2 + f_z(x_0,y_0,z_0)u_3$$

Gradient

The gradient of the the function of three variable *x*, *y*, and *z*

$$\nabla f = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}$$

and, hence, at any point $P(x,y,z)$, the directional derivative of *f* in the dirction of a unit vector $\vec{u}$ is

$$D_{\vec{u}} f = \nabla f(x,y,z)\cdot\vec{u}$$

Example 4

Find the directional derivative of $f(x,y,z)=x^2y-yz^3+z$ at the point $(1,-2,0)$ in the direction of the vector $\vec{a}=2\vec{i}+\vec{j}-2\vec{k}$.

Solution

We can find

$$f_x(x,y,z)=2xy,\ f_y(x,y,z)=x^2-z^2,\ f_z=1-3yz^2$$

Basic Properties of the gradient

Let f and g be differentiable functions. Then

Constant rule: $\nabla c=\vec{0}$ for any constant *c*

Linearity rule: $\nabla(af+bg)=a\nabla f+b\nabla g$ for constant *a* and *b*

Product rule : $\nabla(fg)=f\nabla g+g\nabla f$

Quotient rule : $\nabla\left(\dfrac{f}{g}\right)=\dfrac{g\nabla f-f\nabla g}{g^2},\ g\neq 0$

Power rule : $\nabla(f^n)=nf^{n-1}\nabla f$

(The proof are considered exercises)

5 The Total Differential

For a function of one variable, $y=f(x)$, we defined the differential *dy* to be $dy=f'(x)dx$. For the two-variable case, we make the following analogous definition.

Definition

The **total differential** of the function $f(x,y)$ is

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=f_x(x,y)dx+f_y(x,y)dy$$

where *dx* and *dy* are independent variables. Similarly, for a function of three variables $w=f(x,y,z)$ the **total differential** is

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

Example

Determine the total differential of the given functions:

a. $f(x,y,z)=2x^3+5y^4-6z$ **b.** $f(x,y)=x^2\ln\left(3y^2-2x\right)$

Solution

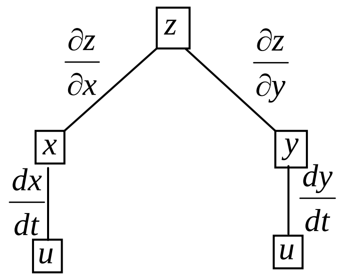
a. $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=6x^2dx+20y^3dy-6dz$

6 Chain Rules

The Chain rule for one independent parameter

Let $f(x,y)$ be a differentiable function of *x* and *y*, and let $x=x(t)$ and $y=y(t)$ to be differentiable functions of *t*. Then $z=f(x,y)$ is a differentiable function of *t*, and

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$



Example 1

Let $z=x^2+y^2$, where $x=\frac{1}{t}$ and $y=t^2$. Compute $\frac{dz}{dt}$ in two ways:

a. by first expressing *z* explicitly in terms of *t*. **b.** by using the chain rule.

Solution

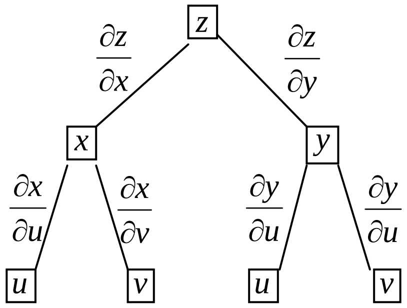
a. By substituting $x=\frac{1}{t}$ and $y=t^2$, we find that

$$z=x^2+y^2=\left(\frac{1}{t}\right)^2+\left(t^2\right)^2=t^{-2}+t^4 \text{ for } t\neq 0$$

Thus, $\frac{dz}{dt}=-2t^{-3}+4t^3$

b. Sine $z=x^2+y^2$ and $x=t^{-1}, y=t^2$, then

$$\frac{\partial z}{\partial x}=2x, \frac{\partial z}{\partial y}=2y, \frac{dx}{dt}=-t^2, \frac{dy}{dt}=2t$$



Use the chain rule for one independent parameter:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=(2x)\left(-t^{-2}\right)+2y(2t)=-2t^{-3}+4t^3$$

Extensions of the Chain Rule

Suppose $z=f(x,y)$ is differentiable at (x,y) and that the partial derivatives of $x=x(u,v)$ and $y=y(u,v)$ exist at (u,v). Then the composite function $z=f[x(u,v),y(u,v)]$ is differentiable at (u,v) with

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \text{ and } \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Example 2

Let $z = 4x - y^2$, where $x = uv^2$ and $y = u^3v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

Solution

First find the partial derivatives

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(4x - y^2\right) = 4 \qquad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(4x - y^2\right) = -2y$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(uv^2\right) = v^2 \qquad \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(u^3v\right) = 3u^2v$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(uv^2\right) = 2uv \qquad \frac{\partial y}{\partial v} = \frac{\partial}{\partial}\left(u^3v\right) = u^3$$

Therefore

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= 4v^2 + (-2y)(3u^2v) \\ &= 4v^2 - 6u^5v^2\end{aligned}$$

EXERCISES

Find the domain of the following function

a $z = \sqrt{1 - x^2 - y^2}$ **b** $z = 1 + \sqrt{-(x-y)^2}$ **c** $z = \ln\left(x^2 + y\right)$

d $z = \ln\left(x + y\right)$ **e** $z = \sqrt{1 - x^2} + \sqrt{1 - y^2}$ **d** $z = \sqrt{x^2 - 4} + \sqrt{4 - y^2}$

Find all first partial derivatives of each of function

1 $f(x, y) = (2x - y)^4$ **2** $f(x, y) = \left(4x - y^2\right)^{3/2}$

3 $f(x, y) = \dfrac{x^2 - y^2}{xy}$ **4** $f(x, y) = e^x \cos y$

5 $f(x, y) = e^y \sin x$ **6** $f(x, y) = \sqrt{x^2 - y^2}$

7 $f(x, y) = e^{xy}$ **8** $f(x, y) = \arctan(4x - 7y)$

9 $f(x, y) = y\cos\left(x^2 + y^2\right)$ **10** $f(r, \theta) = 3r^3 \cos 2\theta$

Verify that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$

11 $f(x, y) = 2x^2y^3 - x^3y^5$ **12** $f(x, y) = \left(x^3 + y^2\right)^5$

13 $f(x, y) = 3e^{2x} \cos y$ **14** $f(x, y) = \arctan xy$

21 If $F(x, y) = \dfrac{2x - y}{xy}$, find $F_x(3, -2)$ and $F_y(3, -2)$.

22 If $F(x, y) = \ln\left(x^2 + xy + y^2\right)$, find $F_x(-1, 4)$ and $F_y(-1, 4)$.

A function of two variables that satisfies **Laplace's Equation** $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ *is said to be* ***harmonic.*** *Show that the following functions are harmonic functions.*

23 $f(x,y)=x^3y-xy^3$

24 $f(x,y)=\ln\left(4x^2+4y^2\right)$

25 If $f(x,y,z)=3x^2y-xyz+y^2z^2$ find each of the following:

a. $f_x(x,y,z)$ **b.** $f_y(0,1,2)$ **c.** $f_{xy}(x,y,z)$

26 If $f(x,y,z)=\left(x^3+y^2+z\right)^4$, find each of the following

a. $f_x(x,y,z)$ **b.** $f_y(0,1,1)$ **c.** $f_{zz}(x,y,z)$

27 The heat equation $c\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$ is the important equation in physics (c is a constant). It is called **partial differential equation.** Show that the functions

$$u=e^{-ct}\sin x \text{ and } u=t^{-1/2}e^{-x^2/(4ct)}$$

satisfy the heat equation.

28 Find the indicated limit or state that it does not exist.

a. $\lim_{(x,y)\to(-1,2)}\frac{xy-y^3}{(x+y+1)^2}$ **b.** $\lim_{(x,y)\to(0,0)}\frac{\sin\left(x^2+y^2\right)}{3x^2+3y^2}$ **c.** $\lim_{(x,y)\to(0,0)}\frac{\tan\left(x^2+y^2\right)}{x^2+y^2}$

d. $\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{x^4-y^4}$ **c.** $\lim_{(x,y)\to(0,0)}\frac{x^4-y^4}{x^2+y^2}$

29 Show that $\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$ does not exit by considering one path to the origin along the *x*-axis and another path along the line $y=x$.

30 Show that $\lim_{(x,y)\to(0,0)}\frac{xy+y^3}{x^2+y^2}$ doesn't exist.

31 Let $f(x,y)=\frac{x^2y}{x^4+y^2}$

a. Show that $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along any straight line $y=mx$.

b. Show that $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along the parabola $y=x^2$.

c. What conclusion do you draw?

Supplementary Exercise

1 Show that the function $z=\varphi\left(x^2+y^2\right)$ satisfies the equation $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0$

2 Find the second order partial derivatives of the function $z=\arctan\frac{x}{y}$

3 Find the total differential of the function of $f(x,y)=2x^2-3xy-y^2$

4 Show that the function $u=\arctan\frac{y}{x}$ satisfies Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Find the first order partial derivatives of the following functions

5 $z = x^2 \sin^2 y$, answer: $\frac{\partial z}{\partial x} = 2x\sin^2 y$, $\frac{\partial z}{\partial y} = x^2 \sin 2y$

6 $z = x^{y^2}$, answer: $\frac{\partial z}{\partial x} = y^2 x^{y^2-1}$, $\frac{\partial z}{\partial y} = x^{y^2} \cdot 2y\ln x$

7 $u = e^{x^2+y^2+z^2}$ answer: $\frac{\partial u}{\partial x} = 2xe^{x^2+y^2+z^2}$, $\frac{\partial u}{\partial y} = 2ye^{x^2+y^2+z^2}$, $\frac{\partial u}{\partial z} = 2ze^{x^2+y^2+z^2}$

8 $u = \sqrt{x^2+y^2+z^2}$

9 $z = \arcsin(x+y)$

Find the total differential of the following functions

10 $z = f(x,y) = x^2 + xy^2 + \sin y$

11 $z = \ln(xy)$

12 $z = e^{x^2+y^2}$

13 Find $f_x(2,3)$ and $f_y(2,3)$ if $f(x,y) = x^2 + y^2$. Answer: $f_x(2,3) = 4,\ f_y(2,3) = 6$

14 Let $f(x,y) = e^{xy^2}$, find f_{xyx}, f_{xxy}, f_{yxx}

15 Find dz/dt using chain rule

a $z = 3x^2y^3, x = t^4, y = t^2$

b $z = \ln(2x^2 + y), x = \sqrt{t}, y = t^{2/3}$

c $z = 3\cos x - \sin xy, x = 1/t, y = 3t$

16 Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ by the chain rule

a $z = 8x^2y - 2x + 3y, x = uv, y = u - v$

b $z = x^2 - y\tan x, x = u/v, y = u^2v^2$

c $z = \frac{x}{y}, x = 2\cos u, y = 3\sin v$

d $z = 3x - 2y, x = u + v\ln u, y = u^2 - v\ln v$

17 Use chain rule to find $\left.\frac{dz}{dt}\right|_{t=3}$ $z = x^2y, x = t^2, y = t+7$

18 Use chain rule to find the value of $\left.\frac{\partial f}{\partial u}\right|_{u=1,v=-2}, \left.\frac{\partial f}{\partial v}\right|_{u=1,v=-2}$ if

$f(x,y) = x^2y^2 - x + 2y, x = \sqrt{u}, y = uv^3$

19 Use chain rule to find the value of

$\left.\frac{\partial z}{\partial r}\right|_{r=2,\theta=\pi/6}$ and $\left.\frac{\partial z}{\partial \theta}\right|_{r=2,\theta=\pi/6}$ if $z = xye^{x/y}, x = r\cos\theta, y = r\sin\theta$.

20 Let $r = \sqrt{x^2+y^2}$, show that

a $\frac{\partial r}{\partial x} = \frac{x}{r}$ **b** $\frac{\partial r}{\partial y} = \frac{y}{r}$ **c** $\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}$ **d** $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$

21 Show that the following function satisfies Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

a $f(x,y)=e^x \sin y+e^y \cos x$

b $f(x,y)=\ln\left(x^2+y^2\right)$

c $f(x,y)=\arctan\frac{2xy}{x^2-y^2}$

22 Find the gradient ∇f

a. $f(x,y)=x^2y+3xy$

b. $f(x,y)=x^3y-y^3$

c. $y=xe^{xy}$

d. $f(x,y)=x^2y\cos y$

e. $f(x,y)=x^2y/(x+y)$

f. $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

g. $f(x,y,z)=x^2y+y^2z+z^2x$

h. $f(x,y,z)=x^2ye^{x-y}$

23 Find the gradient vectors of the given function at the given point

a. $f(x,y)=x^2y-xy^2;(-2,3)$

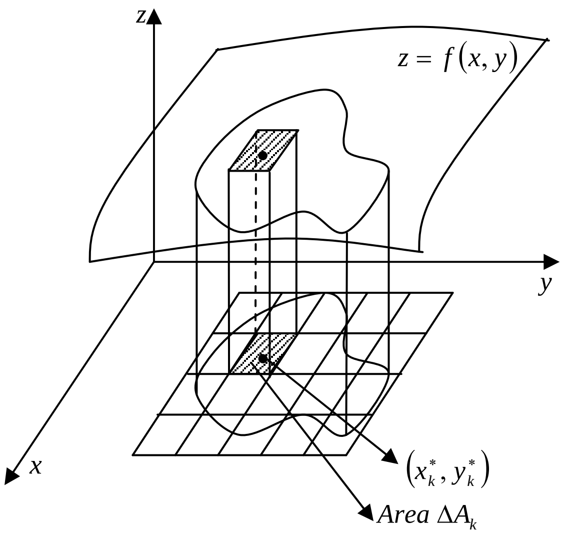
b. $f(x,y)=x^2y+3xy;(2,-2)$

c. $f(x,y)=\frac{x^2}{y};(2,-1)$

Chapter 2

Multiple Integral

1 Double Integral



1.1 Definition

Let R be a region in the *xy*-plane and $f(x,y) \geq 0$. In *xy*-plane, we draw the lines parallel to *x*-axis and *y*-axis so that we get *n* sub-rectangles in the region *R*. The sum of the area of each sub-rectangle is approximate to the area of the region *R*. Any *k*th sub-rectangle whose area is defined by $\Delta A_k = \Delta x \cdot \Delta y$ is the base of a solid with the altitude $f(x_k^*, y_k^*)$. Then the volume of this *k*th solid is defined by

$$V_k = f(x_k^*, y_k^*)\Delta A_k$$

Hence the volume of the solid whose base is the region *R* and bounded above by the function $f(x,y)$ is approximate to

$$V = \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

This sum is called Riemann sums, and the limit of the Riemann sums is denoted by

$$\iint_R f(x,y)dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

which is called double integral of $z = f(x,y)$ over the region *R*.

1.2 Properties of double integrals

(i). *Linearity rule:* for constants *a* and *b*

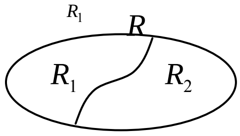
$$\iint_R [af(x,y) + bg(x,y)]dA = a\iint_R f(x,y)dA + b\iint_R g(x,y)dA$$

(ii). *Dominance rule:* if $f(x,y) \geq g(x,y)$ throughout a region *R*, then

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$$

(iii). *Subdivision rule:* If the region R is subdivided in two R_1 and R_2, then

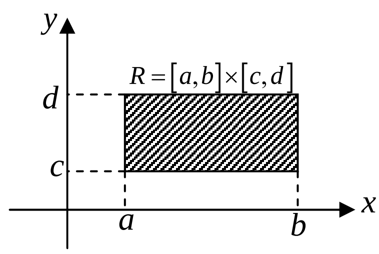
$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$



1.3 The Computation of Double Integral

(i) Over Rectangular Region

If $f(x,y)$ is continuous over therectangle $R: a \leq x \leq b, c \leq y \leq d$, then the double integral $\iint_R f(x,y)dA$ may be evaluated by either iterated integral; that is,



$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Example 1

compute $\iint_R (2-y)dA$, where R is the rectangle with vertices $(0,0),(3,0),(3,2)$ and $(0,2)$. Answer: 6

Example 2

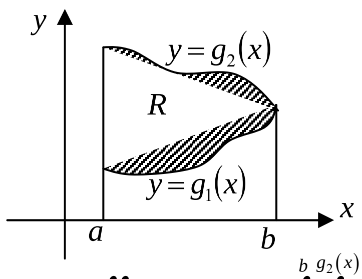
Evaluate $\iint_R x^2 y^5 dA$ where R is the rectangle $1 \leq x \leq 2, 0 \leq y \leq 1$. Answer: $\dfrac{7}{18}$

(ii) Over Nonrectangular Regions

Type I region

This region can be described by the inequalities

$R: a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$



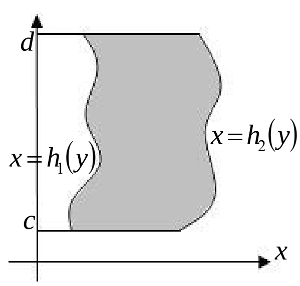
$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Type II region

This region can be described by the inequalities

$R: c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$



Example 3

Evaluate the double integral $\iint_T (x+y)dA$ where T is the triangular region enclosed by the lines $y=0, y=2x, x=1$. Answer: $\frac{4}{3}$.

Example 4

Evaluate $\iint_R xydA$ over the region R enclosed between $y=\frac{1}{2}x, y=\sqrt{x}, x=2, x=4$.

Answer: $\frac{11}{6}$

1.4 Change of Variables in Double Integrals

Let Δ and R be the regions in xy-plane ($\mathbb{R}^2$) where Δ is the new region. A point in this region is defined by (u,v) where $x=x(u,v), y=y(u,v)$. If $z=f(x,y)$ is continuous over the region R, then

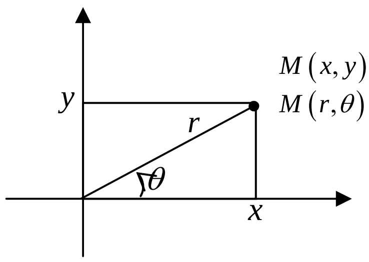
$$\iint_R f(x,y)dxdy = \iint_\Delta f\left[x(u,v), y(u,v)\right]|J|dudv$$

where J is called **Jacobian** and is defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Special Case: *Change from Cartesian Coordinates to Polar Coordinates*

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



then,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence we obtain

$$\iint_R f(x,y)dxdy = \iint_\Delta f(r\cos\theta, r\sin\theta)rdrd\theta$$

Example 5

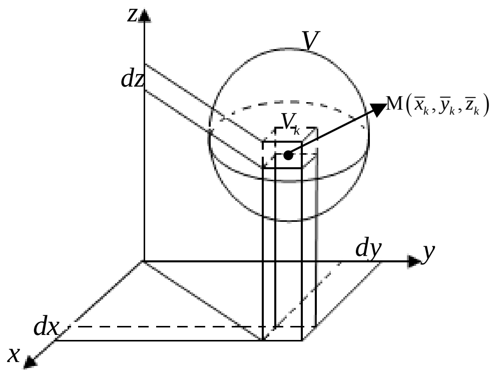
Compute $I=\iint_R (x^2+y^2)dxdy$, R is a region defined by hemi circle $x^2+y^2=2ax, a>0, y\ge 0$. Answer: $\frac{3a^4\pi}{4}$

Example 6

Compute $I=\iint_R (x^2+y^2)dxdy$, $R=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1, y\ge 0\}$ .Answer: $\frac{\pi}{4}$

2 Triple Integral

2.1 Definition



Let V be a solid and $f(x, y, z)$ be a three-variable function defined in $\mathbb{R}^3$. Every plane parallel to each of the three coordinate planes cut the solid V in to n small parallelepipeds, say, v_k (see the figure) whose volume is defined by $v_k = dxdydz$.
Then the triple integral of the function $f(x, y, z)$ over the solid region V is defined as follows:

$$\iiint_V f(x, y, z)dxdydz = \lim_{n\to+\infty} \sum_{k=1}^{n} f(\overline{x}_k, \overline{y}_k, \overline{z}_k)\Delta v_k$$

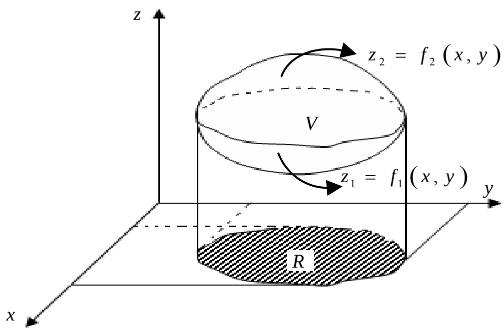
2. 2 The Computation of Triple Integrals

Let V be a parallelepiped defined by the inequality $a \le x \le b, c \le y \le d, m \le z \le n$ then

$$\iiint_V f(x, y, z)dV = \int_m^n \int_c^d \int_a^b f(x, y, z)dxdydz$$

Example 1
Compute $\iiint_V z^2 y e^x dV$ where $V = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le 1, 1 \le y \le 2, -1 \le z \le 1\}$.
Ans: $e-1$
Example 2
Compute $\iiint_V 8xyzdV$ where $V = [0,1]\times[0,2]\times[1,3]$. Answer: 32

2.3 A z-Simple Region



Suppose V is a solid region that is bounded above by the surface $z_1 = f_1(x, y)$ and below by the surface $z_2 = f_2(x, y)$. The projection of this solid region on xy-plane results in region R in the plane. If $w = f(x, y, z)$ is a continuous function over the solid region V, then we obtain

$$\iiint_V f(x, y, z)dV = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} (x, y, z)dz \right) dA$$

Example 3
Compute $\iiint_V xdV$ where V is a solid in the first octant and bounded by

cylinder $x^2 + y^2 = 4$ and the plane $2y + z = 4$. Answer: $\dfrac{20}{3}$.

2.4 Changes of variables in Triple Integrals

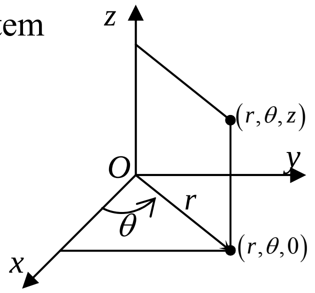
Let V' be a new solid region, and $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$. Then

$$\iiint_V f(x,y,z)\,dV = \iiint_{V'} f\left[x(u,v,w), y(u,v,w), z(u,v,w)\right]|J|\,dudvdw$$

where J is called **Jacobian** and is defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

2.4.1 From Cartesian to Cylindrical Coordinate System



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

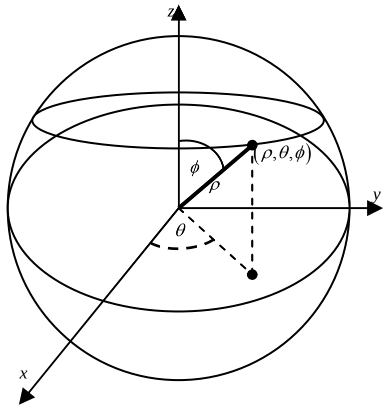
The **Jacobian** is defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

2.4.2 From Cartesian coordinate to Spherical Coordinate System

We have $x = r\cos\theta$, $y = r\sin\theta$ and $r = \rho\sin\phi$, hence we can find

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$



and furthermore we can obtain the relation

$$x^2 + y^2 + z^2 = \rho^2$$

Now we compute the **Jacobian**.

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = -\rho^2\sin\phi$$

Hence $|J| = \rho^2\sin\phi$

Example 4

Compute $\iint_R \sqrt{x^2+y^2}\,dxdy$, where R is the region in xy-plane, enclosed between the circle $x^2+y^2=4$ and $x^2+y^2=9$. Answer: $\frac{38\pi}{3}$

Example 5

Compute $\iiint_V dxdydz$ where V is a solid above xy-plane, inside the

x

cylinder $x^2+y^2=a^2$, but below the parabola $z=x^2+y^2$. Answer: $\dfrac{\pi}{2}a^4$

Example 6

Compute $\iint_R dxdy$ where R is the region enclosed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

(Hint: let $u=\dfrac{x}{a}, v=\dfrac{y}{b}$. Answer: πab)

Example 7

Use spherical coordinate system to compute

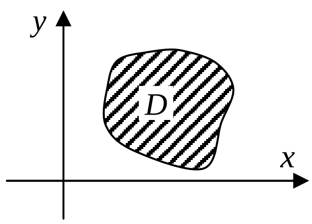
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dzdydx \quad \text{Answer: } \frac{64\pi}{9}$$

3 Applications

3.1 Computation of a Plane Area

The area of the region D in $\mathbb{R}^2$ is found by

$$A=\iint_D dxdy$$



Example 1

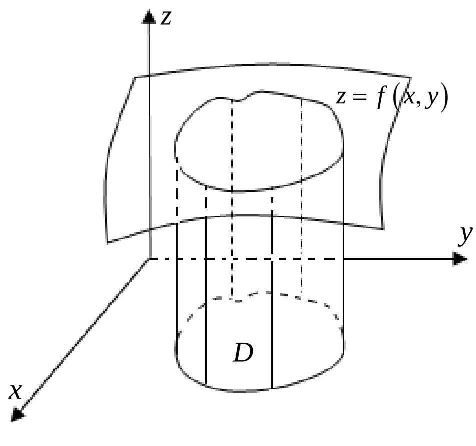
Find the area of the region enclosed between $y=\cos x$ and $y=\sin x$ where $0\le x\le \frac{\pi}{4}$.

Answer: $\sqrt{2}-1$.

3.2 The Volume of a Solid

(i). **The volume** of a solid defined by the surface $z=f(x,y)$, xy-plane where $(x,y)\in D$ (see the figure), $f(x,y)\ge 0$ and $f(x,y)$ is continuous over D, **is found by**

$$V=\iint_D f(x,y)\,dxdy$$



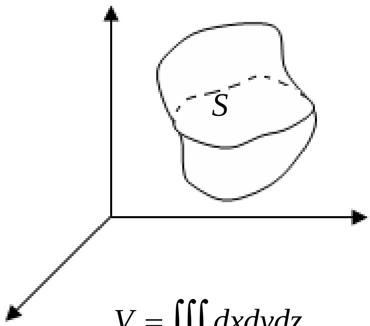
Example 2

Find the volume of a tetrahedron generated by the three coordinate planes and the plane $z=4-4x-2y$. Answer: $\frac{4}{3}$.

Example 3

Find the volume of the solid generated by $z = 2x^2 + y^2 + 1$, $x + y = 1$ and the three coordinate planes. Answer: $\frac{3}{4}$

(ii). Le S be the solid region in the space $\mathbb{R}^3$, then the volume of this solid can be found by



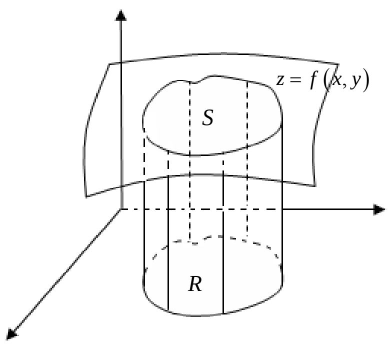
$$V = \iiint_S dxdydz$$

Example 4

Find the volume of a solid enclosed in sphere $x^2 + y^2 + z^2 = 4$ and paraboloid $x^2 + y^2 = 3z$. Answer: $\dfrac{19\pi}{6}$

3.3 Surface Area as a Double Integral

Assume that the function $f(x, y)$ has continuous partial derivatives f_x and f_y in a region R of the *xy*-plane. Then the portion of the surface $z = f(x, y)$ that lies over R has **surface area** S and is found by



$$S = \iint_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA$$

Example 5

Find the surface area of the portion of the plane $x + y + z = 1$ that lies in the first octant (where $x \geq 0, y \geq 0, z \geq 0$)

Solution

Let $z = f(x, y) = 1 - x - y$, then $f_x(x, y) = -1$ and $f_y(x, y) = -1$. Then

$$S = \int_0^1 \int_0^{1-x} \sqrt{(-1)^2 + (-1)^2 + 1}\, dydx = \sqrt{3} \int_0^1 \int_0^{1-x} dydx = \frac{\sqrt{3}}{2}$$

Example 6

Find the surface area of that part of the paraboloid $x^2 + y^2 + z = 5$ that lies above the plane $z = 1$.ans: $\frac{\pi}{6}\left(17^{3/2} - 1\right)$

Example 7

Find the surface area of the portion of the cylinder $y^2 + z^2 = 9$ that lies above the rectangle $R = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 2, -3 \le y \le 3\}$. Answer: 6π

3.4 Mass and Center of Mass

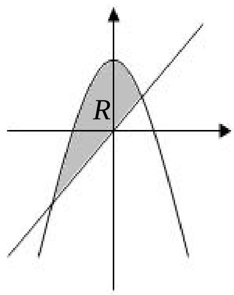
3.4.1 Mass of a Planar Lamina of Variable Density

A planar **lamina** is a flat plate that occupies a region *R* in the plane and is so thin that it can be regarded as two dimensional.

If δ is a continuous density function on the lamina corresponding to a plane region *R*, then the mass *m* of the lamina is given by

$$m = \iint_R \delta(x, y)dA$$

Example 8



Find the mass of the lamina of density $\delta(x, y) = x^2$ that occupies the region *R* bounded by the parabola $y = 2 - x^2$ and the line $y = x$

Solution

We find the domain of integral. By substitution we have

$x = 2 - x^2$ Then $x = -2, 1$

Thus,

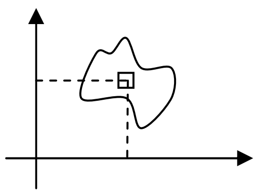
$$m = \iint_R x^2 dA = \int_{-2}^{1} \int_{x}^{2-x^2} x^2 dy dx = \frac{63}{20}$$

Example 9

A triangle lamina with vertices $(0,0), (0,1), (1,0)$ has density function $\delta(x, y) = xy$.

Find its total mass. Answer: $\frac{1}{24}$

3.4.2 Moment and Center of Mass



The *moment* of an object about an axis measures the tendency of the object to rotate about that axis. *It is defined as the product of the object's mass and the signed distance from the axis.*

If $\delta(x, y)$ is a continuous density function on a lamina corresponding to a plane region *R*, then the **moments of mass** with respect to the *x*-axis and *y*-axis, respectively, defined by

$$M_x = \iint_R y\delta(x, y)dA \quad \text{and} \quad M_y = \iint_R x\delta(x, y)dA$$

Furthermore, if *m* is the mass of the lamina, the **center of mass** is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

If the density δ is constant, the point $(\bar{x}, \bar{y})$ is called the **centroid** of the region.

Example 10

Locate the center of mass of the lamina of density $\delta(x,y)=x^2$ that occupies the region R bounded by the parabola $y=2-x^2$ and line $y=x$.

Solution

We have
$$M_x=\iint_R y\delta(x,y)dA=\int_{-2}^{1}\int_{x}^{2-x^2} yx^2\,dydx=-\frac{9}{7}$$
$$M_y=\iint_R x\delta(x,y)dA=\int_{-2}^{1}\int_{x}^{2-x^2} x^3\,dydx=-\frac{18}{5}$$

From previous example we found mass $m=\dfrac{63}{20}$. Then,

$$\bar{x}=\frac{M_y}{m}=\frac{-\frac{18}{5}}{\frac{63}{20}}=-\frac{8}{7} \qquad \bar{y}=\frac{M_x}{m}=\frac{-\frac{9}{7}}{\frac{63}{20}}=-\frac{20}{49}$$

Hence the center of mass is $\left(-\dfrac{8}{7},-\dfrac{20}{49}\right)$

We can use the triple integral to find the mass and center of mass of a solid in $\mathbb{R}^3$ with density $\delta(x,y,z)$. The mass *m*, moments M_{yz}, M_{xz}, M_{xy} about the *yz, xz,* and *xy*-plans, respectively, and coordinates $\bar{x},\bar{y},\bar{z}$ of the center of mass are given by:

Mass $\quad m=\iiint_R \delta(x,y,z)dV$

Moments $\quad M_{yz}=\iiint_R x\delta(x,y,z)dV$, x is the distance to the *yz*-plane

$M_{xz}=\iiint_R y\delta(x,y,z)dV$, y is the distance to the *xz*-plane

$M_{xy}=\iiint_R z\delta(x,y,z)dV$, z is the distance to the *xy*-plane

Center of mass $\quad (\bar{x},\bar{y},\bar{z})=\left(\dfrac{M_{yz}}{m},\dfrac{M_{xz}}{m},\dfrac{M_{xy}}{m}\right)$

Example 11

A solid tetrahedron has vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and constant density $\delta=6$. Find the centroid.

Solution

The tetrahedron can be described as the region in the first octant that lies beneath the plane $x+y+z=1$. Then

$$m=\iiint_R \delta dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y} 6\,dzdydx=1$$

Then we find that

$$M_{yz} = \iiint_R 6x dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 6x dz dy dx = \frac{1}{4}$$

$$M_{xz} = \iiint_R 6y dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 6y dz dy dx = \frac{1}{4}$$

$$M_{xy} = \iiint_R 6z dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 6z dz dy dx = \frac{1}{4}$$

Thus, $\bar{x} = \frac{M_{yz}}{m} = \frac{1}{4}, \bar{y} = \frac{M_{xz}}{m} = \frac{1}{4}, \bar{z} = \frac{M_{xy}}{m} = \frac{1}{4}$

3.5 Moments of Inertia

In general, a lamina of density $\delta(x, y)$ covering region *R* in the first quadrant of the plane has first moment about a line *L* given by the integral $M_L = \iint_R s dm$ where $dm = \delta(x, y) dA$ and $s = s(x, y)$ is the distance from a typical point $P(x, y)$ in *R* to *L*.

Similarly, the second moment of moment of inertia of *R* about *L* is defined by

$$I_L = \iint_R s^2 dm .$$

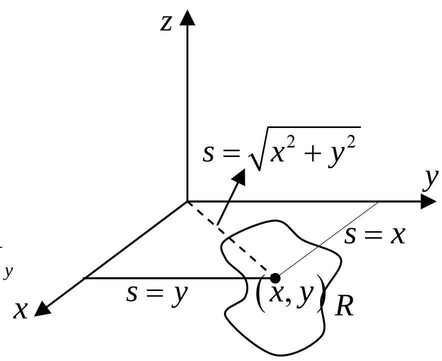
In physics, the moments of inertia measure the tendency of the lamina to resist a change in rotational motion about axis *L*.

The moments of inertia of a lamina of density δ covering the plane region *R* about *x*-, *y*-, *z*-axis, respectively, are given by

$$I_x = \iint_R y^2 \delta(x, y) dA$$

$$I_y = \iint_R x^2 \delta(x, y) dA$$

$$I_z = \iint_R (x^2 + y^2)\delta(x, y) dA = I_x + I_y$$



Example 12

A lamina occupies the region *R* in the plane that is bounded by the parabola $y = x^2$ and the lines $x = 2$, and $y = 1$. The density of the lamina at each point (x, y) is $\delta(x, y) = x^2 y$. Find the moments of inertia of the lamina about the *x*-axis and *y*-axis.

Solution

$$I_x = \iint_R y^2 dm = \iint_R y^2 \delta(x, y) dA = \int_1^2\int_1^{x^2} y^2 \cdot x^2 y dy dx$$

$$= \int_1^2\int_1^{x^2} x^2 y^3 dy dx = \frac{1516}{33}$$

$$I_y = \iint_R x^2 dm = \iint_R x^2 \cdot x^2 y dA = \int_1^2\int_1^{x^2} x^4 y dy dx = \frac{1138}{45}$$

Example 13

A lamina with density $\delta(x,y) = xy$ is bounded by the *x*-axis, the line $x = 8$ and the curve $y = x^{2/3}$. Find the moment of inertia about the 3 axes.

Solution

$$I_x = \iint_R xy^3 dA = \int_0^8 \int_0^{x^{2/3}} xy^3 dydx = \frac{6144}{7}$$

$$I_y = \iint_R x^3 y dA = \int_0^8 \int_0^{x^{2/3}} x^3 y dydx = 6144$$

$$I_z = I_x + I_y$$

We can calculate the moments of inertia of the solid in $\mathbb{R}^3$ which are defined as follows:

$$I_x = \iiint_R \left(y^2 + z^2\right)\delta(x,y,z)dV$$

$$I_y = \iiint_R \left(x^2 + z^2\right)\delta(x,y,z)dV$$

$$I_z = \iiint_R \left(x^2 + y^2\right)\delta(x,y,z)dV$$

Example 14

Find the moment of inertia about the *z*-axis of the solid tetrahedron *S* with vertices $(0,0,0),(0,1,0),(0,0,1),(1,0,0)$ and density $\delta(x,y,z) = x$

Solution

$$I_z = \iiint_R \left(x^2 + y^2\right)\delta(x,y,z)dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\left(x^2 + y^2\right)dzdydx$$

$$= \frac{1}{90}$$

Exercises

1 Let $R = \{(x,y) : 1 \le x \le 4, 0 \le y \le 2\}$. Evaluate $\iint_R f(x,y)\,dA$ where

a. $f(x,y) = \begin{cases} 2, & 1 \le x < 3, 0 \le y \le 2 \\ 3, & 3 \le x \le 4, 0 \le y \le 2 \end{cases}$ (Answer: 14)

b. $f(x,y) = \begin{cases} 2, & 1 \le x < 3, 0 \le y < 1 \\ 1, & 1 \le x < 3, 1 \le y \le 2 \\ 3, & 3 \le x \le 4, 0 \le y \le 2 \end{cases}$ (Answer: 12)

2 Evaluate the following integral

a. $\int_0^1 \int_1^2 xy^2 dxdy$ (Answer: $\frac{1}{2}$)

b. $\int_0^1 \int_0^1 ye^{xy} dxdy$ (Answer: $e - 2$)

c. $\int_0^{1/2}\int_0^{\pi/2} y\sin xy\,dx\,dy$ (Answer: $\frac{1}{20}$)

d. $\int_{-1}^{1}\int_{-1}^{1} x^2\,dx\,dy$ (Answer: $\frac{4}{3}$)

e. $\int_0^1\int_{-1}^{1}\left(xy^2-x^2y\right)dx\,dy$ (Answer: $-\frac{1}{3}$)

f. $\int_1^2\int_0^1 \frac{x}{y}dx\,dy$ (Answer: $\frac{\ln 2}{2}$)

g. $\int_0^1\int_1^2\left(x^2+y^2\right)dx\,dy$ (Answer: $\frac{8}{3}$)

h. $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}dy\,dx$ (Answer: 2)

i. $\int_0^1\int_0^1 \frac{x}{\left(xy+1\right)^2}dy\,dx$ (Answer: $1-\ln 2$)

j. $\int_0^{\ln 2}\int_0^1 xye^{y^2x}dy\,dx$ (Ans: $\frac{1}{2}\left(1-\ln 2\right)$)

3 Evaluate the double integral over the rectangular region *R*.

a. $\iint_R x\sqrt{1-x^2}\,dA$ $\quad R=\left\{(x,y):0\le x\le 1, 2\le y\le 3\right\}$ (Answer: $\frac{1}{3}$)

b. $\iint_R \cos\left(x+y\right)dA$ $\quad R=\left\{(x,y):-\pi/4\le x\le \pi/4, 0\le y\le \pi/4\right\}$ (Answer: 1)

c. $\iint_R 4xy^3\,dA$ $\quad R=\left\{(x,y):-1\le x\le 1,-2\le y\le 2\right\}$ (Answer: 0)

4 Evaluate the following integrals

a. $\int_0^1\int_{x^2}^{x} xy^2\,dy\,dx$ (Answer: $\frac{1}{40}$)

b. $\int_0^3\int_0^{\sqrt{9-y^2}} y\,dx\,dy$ (Answer: 9)

c. $\int_{\sqrt{\pi}}^{\sqrt{2\pi}}\int_0^{x^3} \sin\frac{y}{x}\,dy\,dx$ (Answer: $\frac{\pi}{2}$)

d. $\int_{\pi/2}^{\pi}\int_0^{x^2} \frac{1}{x}\cos\frac{y}{x}\,dy\,dx$ (Answer: 1)

e. $\int_0^a\int_0^{\sqrt{a^2-x^2}} \left(x+y\right)dy\,dx$ (Answer: $\frac{2a^3}{3}$)

f. $\int_0^1\int_0^x y\sqrt{x^2-y^2}\,dy\,dx=\frac{1}{12}$

5 $\iint_R 6xy\,dA$, where *R* is the region bounded by $y=0, x=2,$ and $y=x^2$ (Ans: 32)

6 $\iint_R x\cos xy\,dA$, where *R* is the region enclosed by

$x=1, x=2, y=\pi/2,$ and $y=2\pi/x$. (Answer: $\frac{-2}{\pi}$)

7 $\iint_R x^2\,dA$ where *R* is the region bounded by $y=16/x$, $y=x$ and $x=8$. (Ans: 576)

8 $\iint_R x\left(1+y^2\right)^{-1/2}dA$, where *R* is the region in the first quadrant, enclosed by

$y=4$ and $y=x^2$. (Answer: $\frac{1}{2}\left[\sqrt{17}-1\right]$)

9 $\iint_R \frac{1}{1+x^2}dA$, where *R* is a triangular region with vertices $\left(0,0\right),\left(1,1\right)$ and $\left(0,1\right)$.

(Answer: $\frac{\pi}{4}-\frac{1}{2}\ln 2$)

10 Compute $\iint\limits_R (x-1)\,dA$, R is the region enclosed between $y = x$ and $y = x^3$.

(Answer: $\frac{-1}{2}$)

11 $\int_0^{\frac{\pi}{2}}\int_0^{\sin x} r\cos\theta\, dr d\theta$ (Answer: $\frac{1}{6}$)

12 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\sin\theta} r^2 dr d\theta$ (Answer: 0)

13 $\iint\limits_R e^{-\left(x^2+y^2\right)} dA$, where R enclosed by the circle $x^2 + y^2 = 1$ (Answer: $\pi\left(1-e^{-1}\right)$)

14 $\iint\limits_R \frac{1}{1+x^2+y^2}\,dA$ where R is the sector in the first quadrant that is bounded by $y = 0, y = x$ and $x^2 + y^2 = 4$. (Answer: $\frac{\pi}{8}\ln 5$).

15 $\int_0^1\int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dy dx$ (Answer: $\frac{\pi}{8}$)

16 $\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy dx$ (Answer: $\frac{16}{9}$)

17 $\int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{dydx}{\left(1+x^2+y^2\right)^{3/2}}, (a>0)$ (Answer: $\left(1-\frac{1}{\sqrt{1+a^2}}\right)\frac{\pi}{2}$

18 $\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}} dx dy$ (Answer: $\frac{\pi}{4}\left(\sqrt{5}-1\right)$

19 $\int_0^1\int_{4x}^4 e^{-y^2} dy dx$ (Answer: $\frac{1}{8}\left(1-e^{-16}\right)$)

20 $\iint\limits_R \sin\left(y^3\right) dA$, R is the region bounded by $y = \sqrt{x}, y = 2, x = 0$ (Answer: $\frac{1}{3}\left(1-\cos 8\right)$)

21 $\int_{-1}^1\int_0^2\int_0^1 \left(x^2+y^2+z^2\right) dx dy dz$ (Answer: 8)

22 $\int_0^2\int_{-1}^{y^2}\int_1^z yz\, dx dz dy$ (Answer: 7)

23 $\int_0^3\int_0^{\sqrt{9-z^2}}\int_0^x xy\, dy dx dz$ (Answer: $\frac{81}{5}$)

24 $\int_0^2\int_0^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} x\, dz dy dx$ (Answer: $\frac{128}{15}$)

25 Use spherical coordinate to compute $\int_{-1}^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} e^{-\left(x^2+y^2+z^2\right)^{3/2}} dz dy dx$

(Answer: $\frac{\pi}{3}\left(1-e^{-1}\right)$)

26 Use spherical coordinate to compute $\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz dy dx$

(Answer: 81π)

27 Use double integral to find the volume of the solid tetrahedron that lies in the first octant that is bounded by the three coordinate planes and the plane $z = 5 - 2x - y$. (Answer: $\frac{125}{12}$)

28 Find the volume of the solid that is bounded above by the plane $z = x + 2y + 2$ below by the xy-plane and laterally by $y = 0$ and $y = 1 - x^2$. (Answer: $\frac{56}{15}$)

29 Use double integral to find the volume of the solid that is bounded above by the paraboloid $z = 9x^2 + y^2$, below by the plane $z = 0$ and laterally by the planes $z = 0, y = 0, x = 3$ and $y = 2$. (Answer: 170)

30 Use double integral to find the volume of the wedge cut from the cylinder $4x^2 + y^2 = 9$ by the plane $z = 0$ and $z = y + 3$. (Answer: $\frac{27\pi}{2}$)

31 Use double integral in the first octant bounded by the three coordinate planes and the plane $x + 2y = 4$ and $x + 8y - 4z = 0$ (Answer: $\frac{20}{3}$)

32 Find the volume of the solid bounded above by the paraboliod $z = 1 - x^2 - y^2$ and below by the xy-plane. (Answer: $\frac{\pi}{2}$)

33 Use double integral to find the volume of the solid common to the cylinders $x^2 + y^2 = 25$ and $x^2 + z^2 = 25$. (Answer: $\frac{2000}{3}$)

34 Find the volume of the solid enclosed by the sphere $x^2 + y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 1$. (Answer: $\frac{4\pi}{3}\left(27 - 8^{3/2}\right)$

35 Volume of the solid that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 2y$. (Answer: $\frac{32}{9}$)

36 The integral $\int_0^{+\infty} e^{-x^2} dx$ which arises in probability theory, can be evaluated using a trick. Let the value of the integral be *I*. Thus

$I = \int_0^{+\infty} e^{-x^2} dx = \int_0^{+\infty} e^{-y^2} dy$ since the letter used for the variable of integration in a definite integral does not matter,

a. Show that $I^2 = \int_0^{+\infty} \int_0^{+\infty} e^{-\left(x^2 + y^2\right)} dxdy$

b. Evaluate I^2 by converting to polar coordinate and find I.

37 Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$. (Answer: $\frac{\pi}{6}\left(5\sqrt{5} + 1\right)$)

38 Find the surface area of the portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and parabola $y = x^2$. (Answer: $\frac{\sqrt{5}}{6}$)

39 Find the surface area of the portion of the paraboloid $z = 1 - x^2 - y^2$ that is above the *xy*-plane. (Answer: $\frac{\pi}{6}\left(5\sqrt{5}+1\right)$)

40 Find the surface area of the portion of the surface $z = xy$ that is above the sector in the first quadrant bounded by the line $y = x/\sqrt{3}$, $y = 0$ and the circle $x^2 + y^2 = 9$. (Answer: $\frac{\pi}{18}\left(10\sqrt{10}-1\right)$

41 Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 16$ between the plane $z = 1$ and $z = 2$. (Answer: 8π)

42 Find the surface area of the portion of $x^2 + z^2 = 16$ that lies inside the circular cylinder $x^2 + y^2 = 16$. (Answer: 128)

43 Compute $\iiint\limits_G xy\sin yz dV$, G is the rectangular box defined by the inequalities $0 \le x \le \pi, 0 \le y \le 1, 0 \le z \le \pi/6$. (Answer: $(\pi - 2)\pi/2$)

44 Use triple integral to find the volume of the solid in the first octant bounded by the coordinate planes and the plane $3x + 6y + 4z = 12$. (Answer: 4).

45 Use triple integral to find the volume of the solid bounded by the surface $y = x^2$ and planes $x + z = 4$ and $z = 0$. (Answer: $256/15$).

46 Use triple integral to find the volume of the solid enclosed between the elliptic cylinder $x^2 + 9y^2 = 9$ and the planes $z = 0$ and $z = x + 3$. (Answer: 9π)

47 Use triple integral to find the volume of the solid bounded by the paraboloid $z = 4x^2 + y^2$ and parabolic cylinder $z = 4 - 3y^2$. (Answer: 2π).

48 Use triple integral to find the volume of the solid that is enclosed between the sphere $x^2 + y^2 + z^2 = 2a^2$ and the paraboloid $az = x^2 + y^2$. (Answer: $\frac{\pi}{6}a^3\left(8\sqrt{2}-7\right)$).

49 Let *G* be the tetrahedron in the first octant bounded by the coordinate planes and the planes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, (a > 0, b > 0, c > 0)$.

a. List six different iterated integrals that represent the volume of *G*.

b. Evaluate any one of the six to show that the volume of *G*. (Answer: $\frac{1}{6}abc$).

50 A lamina with density $\delta(x, y) = x + y$ is bounded by the *x*-axis, the line $x = 1$ and the curve $y = \sqrt{x}$. Find its mass and center of mass. ($m = \frac{13}{20}, (\bar{x}, \bar{y}) = \left(\frac{190}{273}, \frac{6}{13}\right)$)

51 A lamina with density $\delta(x, y) = xy$ is in the 1st quadrant and is bounded by the circle $x^2 + y^2 = a^2$ and coordinates axes. Find the mass and center of mass. ($m = \frac{a^4}{8}, (\bar{x}, \bar{y}) = \left(\frac{8a}{15}, \frac{8a}{15}\right)$

52 A triangular lamina is bounded by $y = x$ and $x = 1$, and *x*-axis. Its density is $\delta = 1$. Find centroid of the lamina. ($(\bar{x}, \bar{y}) = \left(\frac{2}{3}, \frac{1}{3}\right)$)

53 A lamina of density 1 occupies the region above *x*-axis and between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ ($a<b$). Answer:

$$m=\frac{\pi}{2}\left(b^2-a^2\right)\ \left(\overline{x},\overline{y}\right)=\left(0,\frac{4\left(b^3-a^3\right)}{3\pi\left(b^2-a^2\right)}\right)$$

54 A cube is defined by the three inequalities $0\le x\le a, 0\le y\le a, 0\le z\le a$, has density $\delta\left(x,y,z\right)=a-x$. Find its mass and center of mass. Answer:

$$m=\frac{a^4}{2}\ \left(\overline{x},\overline{y},\overline{z}\right)=\left(\frac{a}{3},\frac{a}{2},\frac{a}{2}\right).$$

Chapter 3

Ordinary Differential Equation I

1 Introduction

1.1 What is a differential Equation?

A differential equation is any equation, which contains derivatives, either ordinary derivatives or partial derivatives. If the unknown function depends on a single real variable, the differential equation is called an **ordinary differential equation**. The followings are the ordinary differential equations.

$$\frac{dy}{dx}+y=y^2, \quad \frac{d^2y}{dx^2}=xy, \qquad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-4\right)y=0$$

In the differential equations, the unknown quantity $y=y(x)$ is called the **dependent variable**, and the real variable, x, is called the **independent variable**.

In here we define $\frac{dy}{dx}=y', \quad \frac{d^2y}{dx^2}=y'', \quad \frac{d^3y}{dx^3}=y''',\ldots,\frac{d^ny}{dx^n}=y^{(n)}$

1.2 Order of a Differential Equation

The order of a differential equation is **the order of the highest derivative** that occurs in the equation.

For example,

$\frac{dy}{dx}-3y=2$ 1st order

$\frac{d^2y}{dx^2}+x\frac{dy}{dx}-3y=0$ 2nd order

$\frac{d^4y}{dx^4}-y=0$ 4th order

Definition: Ordinary Differential Equation

An n^{th}-order ordinary differential equation is an equation that has the general form

$$F\left(x,y,y',y'',\ldots,y^{(n)}\right)=0$$

where the primes denote differentiation with respect to x, that is,

$y'=\frac{dy}{dx}, y''=\frac{d^2y}{dx^2}$, and so on

1.3 Linear and Nonlinear Differential Equations

A linear differential equation is any differential equation that can be written in the

form $a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)$
with $a_n(x)$ not identical zero. The $a_i(x)$ are known functions of x called **coefficients.** An equation that is not linear is called **nonlinear**. When the coefficients are constant functions, the differential equation is said to have **constant coefficients**. Furthermore, the differential equation is said to be **homogeneous** if $f(x)\equiv 0$ and non-homogeneous if $f(x)$ is *not* identically zero.

Examples of classification of Differential Equations:

Differential equation	Linear or Nonlinear	Order	Homogeneous or non-homogeneous	Constant or variable coefficients
$\frac{dy}{dx}+xy=1$	Linear	1	Non-homogeneous	Variable
$\frac{d^2y}{dx^2}+y\frac{dy}{dx}+y=x$	Nonlinear	2	Non-homogeneous	Variable
$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=0$	Linear	2	Homogeneous	Constant
$\frac{d^4y}{dx^4}+3y=\sin x$	Linear	4	Non-homogeneous	Constant

1.4 Solutions

A function is a **solution** of a differential equation on an interval if, when substituted into the differential equation, the resulting equality is true for all values of x in the domain of $y(x)$.

Example 1

Verify that $y(x)=\sin x+x^2$ is a solution of the second order linear equation

$$y''+y=x^2+2$$

Example 2

Verify that the function $y(x)=3e^{2x}$ is a solution of the differential equation $\frac{dy}{dx}-2y=0$ for all x.

1.5 Implicit/Explicit Solution

An **explicit solution** is any solution that is given in the form $y=y(x)$. In some occasions, it is impossible to deduce an explicit representation for y in term of x. Such solutions are called **implicit solutions.**

Example 3

The relation $x = e^y + y$ implicitly defines y as a function of x. Verify that this implicitly defined function is a solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{x - y + 1}$$

Solution

Differentiating $x = e^y + y$ with respect to x gives

$$1 = e^y \frac{dy}{dx} + \frac{dy}{dx}$$

$$1 = \left(e^y + 1\right)\frac{dy}{dx}$$

Thus

$$\frac{dy}{dx} = \frac{1}{e^y + 1}$$

Substitute this and $x = e^x + y$ into the equation gives

$$\frac{1}{e^y + 1} = \frac{1}{\left[e^y + y\right] - y + 1}$$

or

$$\frac{1}{e^y + 1} = \frac{1}{e^y + 1}$$

which is true.

1.6 Initial-Value Problem (IVP)

An **initial-value problem** for an *nth*-order equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$$

consists in finding the solution to the differential equation on an interval I that also satisfies the n **initial conditions** $y\left(x_0\right) = y_0, y'\left(x_0\right) = y_1, \ldots, y^{(n-1)}\left(x_0\right) = y_{n-1}$ where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

Example 4

Verify that $y\left(x\right) = \sin x + \cos x$ is a solution of the initial value problem

$$y'' + y = 0,\ y\left(0\right) = 1, y'\left(0\right) = 1$$

Solution

We have $y'\left(x\right) = \cos x - \sin x$

$y''\left(x\right) = -\sin x - \cos x$

Substituting into the equation gives

$y'' + y = \left(-\sin x - \cos x\right) + \left(\sin x + \cos x\right) = 0$ Hence $y\left(x\right)$ satisfies the differential equation. To verify that $y\left(x\right)$ also satisfies the initial conditions, we observe that

$$y(0)=\sin 0+\cos 0=1,\ y'(0)=\cos 0-\sin 0=1$$

1.7 General Solution of a Differential Equation

In general case when solving an *nth*-order equation $F\left(x, y, y', \ldots, y^{(n)}\right)=0$ we generally obtain **n-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n)=0$. A solution of a differential equation that is free of arbitrary parameters is called a specific or **particular solution.**

Example 5

The function $y(x)=\dfrac{3}{4}+\dfrac{c}{x^2}$ is the **general solution** to $2xy'+4y=3$. From this example the function $y(x)=\dfrac{3}{4}-\dfrac{9}{4x^2}$ is the **particular solution** when applying the initial condition $y(1)=-4$ on the equation $2xy'+4y=3$; that is, it is the solution to the initial value problem $2xy'+4y=3, y(1)=-4$.

Exercises

Show that each function is a solution of the given differential equation. Assume that *a* and *c* are constants.

1. $\dfrac{dy}{dx}=ay$ $\qquad y=e^{ax}$
2. $\dfrac{dy}{dx}=y+e^x$ $\qquad y=xe^x$
3. $\dfrac{d^2y}{dx^2}+a^2y=0$ $\qquad y=c\sin ax$
4. $\dfrac{1}{4}\left(\dfrac{d^2y}{dx^2}\right)^2-x\dfrac{dy}{dx}+y=1-x^2$ $\qquad y=x^2$

Show that the following relation defines an implicit solution of the given differential equation

5. $yy'=e^{2x}$ $\qquad y^2=e^{2x}$
6. $2xyy'=x^2+y^2$ $\qquad y^2=x^2-cx$

Verify that the specified function is a solution of the given initial-value problem

Differential Equation	Initial Condition(s)	Function
1. $y'+y=0$	$y(0)=2$	$y(x)=2e^{-x}$
2. $y'=y^2$	$y(0)=0$	$y(x)=0$
3. $y''+4y=0$	$y(0)=1 \quad y'(0)=0$	$y(x)=\cos 2x$
5. $y''+3y'+2y=0$	$y(0)=0 \quad y'(0)=1$	$y(x)=e^{-x}-e^{-2x}$

2 Separable Equations

A differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called **separable** if it can be written as

$$\frac{dy}{dx} = h(x)g(y)$$

That is, $f(x, y)$ factors into a function of x times a function of y. Either $h(x)$ or $g(y)$ may be constant so that every differential equation of the form

$$\frac{dy}{dx} = h(x) \text{ or } \frac{dy}{dx} = g(y)$$

is separable.

Some examples of such functions are

$$e^{x+y} = e^x \cdot e^y,\ x^2y = x^2 \cdot y$$

$$xy + 2x + y + 2 = (x+1)(x+2)$$

$$3y^3 \text{ (Here } h(x) = 1)$$

If $f(x, y)$ has been factored so that the differential equation is written as in the above examples, then we divide by $g(y)$ to get

$$\frac{1}{g(y)}\frac{dy}{dx} = h(x)$$

Next we anti-differentiate both sides with respect to x

$$\int \frac{1}{g(y)}\frac{dy}{dx}dy = \int h(x)dx$$

By the chain rule $dy = \frac{dy}{dx}dx$ so $\int \frac{1}{g(y)}dy = \int h(x)dx$

Solution by Separation of Variables

To solve $\frac{dy}{dx} = f(x, y)$ by separations of variables, we proceed by the following:

(i). Factor $f(x, y) = h(x)g(y)$

(ii). Rewrite $dy/dx = h(x)g(y)$ in differential form as $\frac{1}{g(y)}dy = h(x)dx$

(iii). The solution is $\int \frac{1}{g(y)}dy = \int h(x)dx$

Example1

Solve the differential equation $\frac{dy}{dx} = y^2 + 1$

Solution

Rewrite the differential equation in differential form as

$$\frac{1}{y^2+1}dy = dx$$

So that

$$\int \frac{1}{y^2+1}dy = \int dx$$

Then

$$Arc\tan y = x + C$$

Hence

$$y = \tan(x+C)$$

Example2

Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y},\ y(0)=1$

In differential form we obtain

$$ydy = -xdx$$

So

$$\int ydy = \int -xdx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C$$

By substitute the initial condition $x=0, y=1$ into this equation gives $C = \frac{1}{2}$.

Hence the implicit solution is $x^2 + y^2 = 1$.

Example3

Solve the differential equation $\frac{dy}{dx} = \frac{1+x+x^3}{2+y^2+y^6}$

Exercises for section 2

Solve the given differential equations

1. $\frac{dy}{dx} = x - x^2$

2. $\frac{dy}{dx} = \frac{2y}{x}$

3. $\frac{dy}{dx} = e^{x+y}$

4. $\frac{dy}{dx} = \frac{2x(y+1)}{y}$

5. $\frac{dy}{dx} = \frac{1}{x-x^3}$

6. $\frac{dy}{dx} = y - y^2$

7. $\frac{dy}{dx} = \frac{x}{y^2\sqrt{1+x^2}}$

8. $\frac{dy}{dx} = x^2 - 2x + 5$

9. $\frac{dy}{dx} = \frac{y}{1+x},\ y(0)=1$

10. $\frac{dy}{dx}=\frac{x+xy^2}{4y},\ y(1)=0$

11. $x\frac{dy}{dx}-y=2x^2y,\ y(1)=e$

12. $\frac{dy}{dx}=y^2-4$

13. $\frac{dy}{dx}=x^2y^2+y^2+x^2+1,\ y(0)=2$

14. $\frac{du}{dt}=\frac{t^2+1}{u^2+4}$

3 First Order Linear Equations

The first order linear equation is of the form

$$a_1(x)\frac{dy}{dx}+a_0(x)y=h(x)$$

with $a_1(x)\neq 0$.

The equation can be rewritten as

$$\frac{dy}{dx}+p(x)y=g(x)$$

where $p(x)=a_0(x)/a_1(x)$ and $g(x)=h(x)/a_1(x)$.

We now solve the later equation. We will try to find a function $u(x)$, called an **integrating factor**, such that

$$u(x)\left(\frac{dy}{dx}+p(x)y\right)=\frac{d(uy)}{dx}$$

To find $u(x)$, we proceed as follows:

$$u(x)y'+u(x)p(x)y=u'(x)y+u(x)y'$$

If we assume that $y(x)\neq 0$, we arrive at

$$u'(x)=p(x)u(x)$$

We can find a solution $u(x)>0$ by separating variables, getting

$$\frac{u'(x)}{u(x)}=p(x)$$

$$\ln u(x)=\int p(x)dx$$

$$u(x)=e^{\int p(x)dx}$$

we have $u(x)\left(\frac{dy}{dx}+p(x)y\right)=\frac{d(uy)}{dx}$ or $u(x)\left(y'+p(x)y\right)=(uy)'$

but

$$u(x)\left(y'+p(x)y\right)=u(x)g(x)$$

then

$$(uy)'=u(x)g(x)\Rightarrow uy=\int u(x)g(x)dx+C$$

Summary of First Order Linear Procedure

- **a.** Rewrite the differential equation as $\frac{dy}{dx}+p(x)y=g(x)$
- **b.** Compute the integrating factor $u=e^{\int p(x)dx}$
- **c.** Multiply both sides by $u(x)$ to get $\frac{d}{dx}(uy)=ug$
- **d.** Anti-differentiate both sides with respect to *x*,

$$uy=\int ug\,dx+C$$

- **e.** If there is initial condition, use it to find *C*.
- **f.** Solve for *y*.

Example1

Find all solutions of $xy'-y=x^2,\quad x>0$

Solution

The equation can be rewritten as

$$y'-\frac{1}{x}y=x$$

then $p(x)=-\frac{1}{x}$. Thus

$$u(x)=e^{-\int\frac{dx}{x}}=e^{-\ln x}=e^{\ln x^{-1}}=x^{-1}$$

Multiplying each side of the differential equation by the integrating factor, we get

$$x^{-1}\left(y'-\frac{1}{x}y\right)=1$$

$$\frac{d}{dx}\left(x^{-1}y(x)\right)=x^{-1}x=1$$

$$x^{-1}y(x)=x+C$$

$$y(x)=x^2+Cx$$

Example2

Solve the initial value problem $\frac{dy}{dx}+\frac{3}{x}y=\frac{\sin x}{x^3},(x>0),y(\pi/2)=1$

Solution

Since $p(x)=3/x$, the integrating factor is

$$\mu(x)=e^{\int(3/x)dx}=e^{3\ln x}=x^3$$

Multiplying both sides of equations by integral factor give

$$x^3\left(y'+\frac{3}{x}y\right)=\sin x$$

$$\frac{d}{dx}\left(x^3y\right)=\sin x$$

by integration we obtain

$$x^3 y = -\cos x + C$$

Thus

$$y(x) = \frac{C}{x^3} - \frac{\cos x}{x^3} \qquad (x > 0)$$

Since $x = \frac{\pi}{2}$, $y = 1$, then $1 = \frac{C}{(\pi/2)^3}$, that is $C = \frac{\pi^3}{8}$

Hence the solution is $y(x) = \frac{\pi^3}{8x^3} - \frac{\cos x}{x^3}, (x > 0)$

4 Bernoulli Equations

A *Bernoulli Equations* is a first-order differential equation in the form

$$\frac{dy}{dx} + p(x) y = q(x) y^n \qquad (1)$$

If $n = 0$ or $n = 1$, then the Bernoulli equation is already first order linear and can be solved by the method of the previous section. If $n \neq 0$ and $n \neq 1$ then the substitution $v = y^{1-n}$ will change the Bernoulli equation to a linear equation in v and x.

Let $v = y^{1-n}$, then

$$v' = (1-n) y^{-n} y' \text{ or } y' = \frac{v'}{(1-n) y^{-n}} = \frac{y^n v'}{1-n}$$

or

$$\frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}$$

Substituting this into (1) yields

$$\frac{y^n}{1-n} \cdot \frac{dv}{dx} + p(x) y = q(x) y^n$$

By dividing both sides by $y^n/(1-n)$ and use $y^{1-n} = v$, we obtain

$$\frac{dv}{dx} + (1-n) p(x) v = (1-n) q(x)$$

which is a linear first-order differential equation in v.

Example1

Solve the differential equation $\frac{dy}{dx} = y + y^3$

Solution

The equation can be rewritten as

$$\frac{dy}{dx} - y = y^3$$

with $n=3, p=-1$ and $q=1$

Let $v=y^{1-3}=y^{-2}=\frac{1}{y^2}$, then $v'=\frac{-\left(y^2\right)'}{\left(y^2\right)^2}=\frac{-2yy'}{y^4}=-\frac{2y'}{y^3}$

So

$$y'=\frac{-y^3v'}{2}$$

By the substitution of y' into the equation, we obtain

$$\frac{-y^3v'}{2}-y=y^3$$

$$\frac{-y^3}{2}\frac{dv}{dx}-y=y^3$$

Dividing both sides by $\frac{-y^3}{2}$ gives

$$\frac{dv}{dx}+\frac{2}{y^2}=-2$$

Substitute $v=\frac{1}{y^2}$ in the equation we obtain

$$\frac{dv}{dx}+2v=-2$$

We have $p(x)=2$ then $u(x)=e^{\int 2dx}=e^{2x}$

Multiplication of the equation by the integral factor gives

$$e^{2x}\left(v'+2v\right)=-2e^{2x}$$

$$\left(ve^{2x}\right)'=-2e^{2x}$$

Anti-differentiating with respect to x gives

$$ve^{2x}=-e^{2x}+C$$

Solving for v, we obtain

$$v=-1+Ce^{-2x}$$

$$\frac{1}{y^2}=Ce^{-2x}-1$$

$$y=\pm\left[Ce^{-2x}-1\right]^{-1/2}$$

Exercises for section 3 and 4

1. $\frac{dy}{dx}+2y=0$

2. $\frac{dy}{dx}+2y=3e^x$

3. $\frac{dy}{dx}-y=e^{3x}$

4. $\frac{dy}{dx}+y=\sin x$

5. $\frac{dy}{dx}+y=\frac{1}{1+e^{2x}}$

6. $\frac{dy}{dx}+2xy=x$

7. $\frac{dy}{dx}+3x^2y=x^2$

8. $\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{x^2}$

9. $x\frac{dy}{dx}+y=2x$

10. $\cos x\frac{dy}{dx}+y\sin x=1$

11. $\frac{dy}{dx}-\frac{2y}{x}=x^2\cos x$

12. $\left(1+e^x\right)\frac{dy}{dx}+e^xy=0$

13. $\left(x^2+9\right)\frac{dy}{dx}+xy=0$

14. $\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y=e^{-2x}$

15. $xy'+y=xy^3$

16. $y'+y=y^2$

17. $y'+y=y^2e^x$

18. $y'+xy=6x\sqrt{y}$

19. $y'+y=y^{-2}$

Solve the initial-value problem

20. $\frac{dy}{dx}-y=1 \quad y(0)=1$

21. $\frac{dy}{dx}+2xy=x^3 \quad y(1)=1$

22. $\frac{dy}{dx}-\frac{3}{x}y=x^3 \quad y(1)=4$

23. $\frac{dy}{dx}+2xy=x \quad y(0)=1$

24. $\left(1+e^x\right)\frac{dy}{dx}+e^xy=0 \quad y(0)=1$

25. $y'+y=\sin x,\ \ y(\pi)=1$

26. $y'+\frac{2}{x}y=-x^9y^5,\ \ y(-1)=2$

27. $y'=y^4-y,\ \ y(0)=1$

5 Riccati Equation

The nonlinear differential equation

$$\frac{dy}{dx}=f(x)+g(x)y+h(x)y^2 \quad (1)$$

is called **Riccati equation.** In order to solve a Riccati equation, one will need a particular solution. Let $y_0(x)$, then the substitution

$$y=y_0(x)+\frac{1}{w(x)}$$

converts the equation to

$$\frac{dw}{dx}+\left[g(x)+2h(x)y_0(x)\right]w+h(x)=0$$

which is a linear differential equation of first order with respect to the function $w=w(x)$.

Proof

We have $y=y_0(x)+\frac{1}{w(x)}$. Differentiating with respect to *x*, yields

$$\frac{dy}{dx}=\frac{dy_0}{dx}-\frac{1}{w^2}\frac{dw}{dx}$$

Substituting *y* and $\frac{dy}{dx}$ into (1), yields

$$\frac{dy_0}{dx}-\frac{1}{w^2}\frac{dw}{dx}=f(x)+g(x)\left[y_0+\frac{1}{w}\right]+h(x)\left[y_0+\frac{1}{w}\right]^2$$

$$\frac{dy_0}{dx}-\frac{1}{w^2}\frac{dw}{dx}=f(x)+g(x)y_0+g(x)\frac{1}{w}+y_0^2h(x)+2y_0\frac{1}{w}h(x)+\frac{1}{w^2}h(x)$$

$$-\frac{1}{w^2}\frac{dw}{dx}=-\frac{dy_0}{dx}+f(x)+g(x)y_0+g(x)\frac{1}{w}+y_0^2h(x)+2y_0\frac{1}{w}h(x)+\frac{1}{w^2}h(x)$$

$$-\frac{1}{w^2}\frac{dw}{dx}=-\frac{dy_0}{dx}+f(x)+g(x)y_0+g(x)\frac{1}{w}+y_0^2h(x)+2y_0\frac{1}{w}h(x)+\frac{1}{w^2}h(x)$$

Since $\frac{dy_0}{dx}=f(x)+g(x)y_0+h(x)y_0{}^2$, we obtain

$$-\frac{1}{w^2}\frac{dw}{dx}=-\frac{dy_0}{dx}+\frac{dy_0}{dx}+g(x)\frac{1}{w}+2y_0\frac{1}{w}h(x)+\frac{1}{w^2}h(x)$$

$$\frac{1}{w^2}\frac{dw}{dx}+g(x)\frac{1}{w}+2y_0\frac{1}{w}h(x)+\frac{1}{w^2}h(x)=0$$

$$\frac{dw}{dx}+\left[g(x)+2y_0h(x)\right]w+h(x)=0$$

Example

Solve the Riccati Equation $\frac{dy}{dx}=-2-y+y^2$, given that $y_0=2$ is a particular solution.

Solution

Substituting $y=2+\frac{1}{w}$ converts the equation to

$$\frac{dw}{dx}+3w=-1$$

which is a first order linear equation. Its integrating factor is

$$\mu=e^{3\int dx}=e^{3x}$$

Multiplying both sides of the equation by integrating factor, yields

$$e^{3x}\left[\frac{dw}{dx}+3w\right]=-e^{3x}$$

$$e^{3x}w=-\int e^{3x}dx=-\frac{1}{3}e^{3x}+c$$

$$w=-\frac{1}{3}+ce^{-3x}$$

Finally the general solution to the equation is

$$y=2+\frac{1}{-\frac{1}{3}+ce^{-3x}}$$

6 Exact Equations

A differential equation

$$M(x,y)dx+N(x,y)dy=0$$

is **exact** if there exists a function $F(x,y)$ such that

$$dF(x,y) = M(x,y)\,dx + N(x,y)\,dy$$

If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous first partial derivatives on some domain, then the equation is exact if and only if

$$M_y = N_x$$

To solve the exact equation, first solve the equations $F_x = M$ and $F_y = N$ for $F(x,y)$. The solution is given implicitly by $F(x,y) = C$, where C represents an arbitrary constant.

Method for Solution of Exact Equations

a. Write the differential equation in the form $M(x,y)\,dx + N(x,y)\,dy = 0$

b. Compute M_y and N_x. If $M_y \neq N_x$, the equation is not exact and this technique will not work. If $M_y = N_x$, the equation is exact and this technique will work.

c. Either anti-differentiate $F_x = M$ with respect to x or $F_y = N$ with respect to y. Anti-differentiating will introduce an arbitrary function of the other variable.

d. Take the result for F from step **c.** and substitute for F to find the arbitrary function.

e. The solution is $F(x,y) = C$.

Example 1

Solve $(2x+1+2xy)\,dx + (x^2+4y^3)\,dy = 0$

Solution

In this problem, $M = 2x+1+2xy$ and $N = x^2+4y^3$. Since $M_y = N_x = 2x$, the equation is exact. Then $F_x = M = 2x+1+2xy, F_y = M = x^2+4y^3$

Either equation can be anti-differentiated. We shall anti-differentiate the second one:

$F = \int F_y dy = \int (x^2+4y^3)dy = x^2y + y^4 + k(x)$**,** where $k(x)$ is the unknown function of x. We then substitute this expression for F in the other equation $F_x = M = 2x+1+2xy$ in other to find $k(x)$.

$$\frac{\partial\left(x^2y + y^4 + k(x)\right)}{\partial x} = 2x+1+2xy$$

Then

$$2xy + k'(x) = 2x+1+2xy$$

or

$$k'(x) = 2x+1 \text{ and } k(x) = x^2 + x$$

Thus $F(x,y) = x^2y + y^4 + x^2 + x$ and the general solution is

$$x^2y + y^4 + x^2 + x = C$$

Example 2

Solve $y' = \dfrac{2+ye^{xy}}{2y-xe^{xy}}$ (The solution is $F(x,y) = 2x + e^{xy} - y^2 = C$)

Some non-exact equations can be made exact by the following procedure.

Integrating Factor Method

For differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$, first compute M_y and N_x

1a. If $(M_y - N_x)/N$ cannot be expressed as a function of *x* only, then we do not have an integrating factor that is a function of *x* only. If $(M_y - N_x)/N = Q(x)$ is a function of *x*, then $u(x) = e^{\int Q(x)}dx$ is an integrating factor.

1b. If $(N_x - M_y)/M$ cannot be expressed as a function of *y* only, then we do not have an integrating factor that is a function of *y* only. If $(M_y - N_x)/N = R(y)$ is a function of *y*, then $u(y) = e^{\int R(y)dy}$ is an integrating factor.

2. Multiply $M(x,y)\,dx + N(x,y)\,dy = 0$ by integrating factor

3. Solve the exact equation $u(x,y)M(x,y)\,dx + u(x,y)N(x,y)\,dy = 0$

Example 3

Solve the differential equation $(3y^2+4x)\,dx + (2xy)\,dy = 0$

Solution

In this example,

$$M = 3y^2 + 4x \qquad N = 2xy$$

so $M_y = 6y, N_x = 2y$ and the equation is not exact. How ever

$$\frac{M_y - N_x}{N} = \frac{6y-2y}{2xy} = \frac{2}{x}$$

is a function of *x*. Thus there is an integrating factor

$$u(x) = e^{\int \frac{2}{x}dx} = e^{(2\ln|x|)} = x^2$$

Multiplying the differential equation by x^2 gives the new differential equation

$$(3x^2y^2 + 4x^3)\,dx + (2x^3y)\,dy = 0$$

which is exact. The solution of this equation can be found to be $F = x^3y^2 + x^4 = C$

Exercises

Solve the differential equation

1. $(y+2xy^3)\,dx + (1+3x^2y^2+x)\,dy = 0$, $y(1) = -5$

2. $e^{x^3}(3x^2y - x^2)\,dx + e^{x^3}dy = 0$, $y(0) = -1$

3. $(y\sin x + xy\cos x)dx + (x\sin x + 1)dy = 0$

4. $(4t^3y^3 - 2ty)dt + (3t^4y^2 - t^2)dy = 0$

5. $(x - y)dx + (x + y)dy = 0$

6. $3x^2y^2dx + (2x^3y + 4y^3)dy = 0$

7. $(x + \sin y)dx + (x\cos y - 2y)dy = 0$

8. $(3x^2 + 2xy + y^3)dx + (x^2 + 3xy^2 + \cos y)dy = 0,\ y(0) = 0$

9. $(\sin y + e^x)dx + (x\cos y - 2y)dy = 0$

10. $x^{-1}ydx + (\ln x + 3y^2)dy = 0$

Find an appropriate integrating factor for each differential equation and then solve

11. $(y + 1)dx - xdy = 0$

12. $ydx + (1 - x)dy = 0$

13. $(x^2 + y + y^2)dx - xdy = 0$

14. $(y + x^3y^3)dx + xdy = 0$

15. $(y + x^4y^2)dx + xdy = 0$

16. $(3x^2y - x^2)dx + dy = 0$

17. $dx - 2xydy = 0$

18. $2xydx + y^2dy = 0$

19. $ydx + 3xdy = 0$

7 Homogeneous Equations

In this section we develop a substitution technique that can sometimes be used when other techniques fail. Suppose that we have the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

and the value of $f(x, y)$ depends only on the ratio $v = y/x$, so that we can think of $f(x, y)$ as a function F of y/x,

$$f(x, y) = F(y/x) = F(v)$$

Examples of such functions are:

1/. $\dfrac{x + 3y}{2x + y} = \dfrac{1 + 3(y/x)}{2 + (y/x)}$ $\qquad F(v) = \dfrac{1 + 3v}{2 + v}$

2/. $e^{y/x}$ $\qquad F(v) = e^v$

3/. $\dfrac{x^2 + y^2}{3xy + y^2} = \dfrac{1 + (y/x)^2}{3(y/x) + (y/x)^2}$ $\qquad F(v) = \dfrac{1 + v^2}{3v + v^2}$

Using F, we can rewrite (1) as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad (2)$$

A differential equation in the form (1) that may be written in the form (2) is sometimes called *homogenous.*

Let $y = xv$ so that (2) becomes

$$\frac{d(xv)}{dx} = f(v) \text{ or } v + x\frac{dv}{dx} = F(v)$$

which may always be solved by separation of variables (separable equation)

$$\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$$

There is an alternative definition of "homogeneous" that is easier to verify:

$$\frac{dy}{dx} = f(x,y)$$

is homogeneous equation if

$$f(tx,ty) = f(x,y) \qquad (3)$$

for all t such that (x,y) and (tx,ty) are in the domain of f.

Definition

A function $f(x,y)$ is said to be **homogeneous** of degree *n* in *x* and *y* if, for every *k*,

$$f(kx,ky) = k^n f(x,y)$$

where *k* is a real parameter.

Example 1

(i). $f(x,y) = x^2 + xy$ is homogeneous of degree 2 since

$$f(kx,ky) = (kx)^2 + (kx)(ky) = k^2(x^2+xy) = k^2 f(x,y)$$

(ii). $f(x,y) = e^{x/y}$ is homogeneous of degree zero since

$$f(kx,ky) = e^{kx/ky} = k^0 e^{x/y}$$

(iii). $f(x,y) = x^2 + y^2 + 5$ is not homogeneous.

Summary of Method for Homogeneous Equations

(i). Verify that the equation is homogeneous.

(ii). Let $y = xv$ to get $x(dv/dx) + v = F(v)$

(iii). Solve the separable equation

(iv). Let $v = y/x$ to get the answer in terms of y and x

Example 2

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{-2x+5y}{2x+y}$

Solution

The equation is homogeneous since

$$f(tx,ty) = \frac{-2tx+5ty}{2tx+ty} = \frac{-2x+5}{2x+y} = f(x,y)$$

Let $y = xv$, the equation becomes

$$v + x\frac{dv}{dx} = \frac{-2x+5xv}{2x+xv} = \frac{-2+5v}{2+v}$$
$$x\frac{dv}{dx} = \frac{-2+5v}{2+v} - v$$
$$x\frac{dv}{dx} = \frac{-2+3v-v^2}{2+v} = \frac{1}{-\frac{2+v}{v^2-3v+2}}$$

By separation of variables

$$\int \frac{2+v}{v^2-3v+2}dv = \int -\frac{1}{x}dx$$
$$\int\left(\frac{4}{v-2} - \frac{3}{v-1}\right)dv = \int -\frac{1}{x}dx$$
$$4\ln|v-2| - 3\ln|v-1| = -\ln|x| + C$$

Taking the exponential of both sides of the last equation yields

$$e^{4\ln|v-2|-3\ln|v-1|} = e^{C-\ln|x|}$$
$$\frac{(v-2)^4}{(v-1)^3} = \frac{C'}{x}$$

where the absolute values have been dropped by allowing C' to take on negative or positive value. Let $v = y/x$ to get

$$(y-2x)^4 = C'(y-x)^3$$

Example 3

Solve the differential equation $y' = \dfrac{2y^4+x^4}{xy^3}$

$f(tx,ty) = \dfrac{2(ty)^4+(tx)^4}{(tx)(ty)^3} = \dfrac{t^4\left(2y^4+x^4\right)}{t^4\left(xy^3\right)} = \dfrac{2y^4+x^4}{xy^3} = f(x,y)$, so the equation is homogeneous.

$$v + x\frac{dv}{dx} = \frac{2(xv)^4+x^4}{x(xv)^3}$$
$$x\frac{dv}{dx} = \frac{v^4+1}{v^3}$$

Separating variables yields

$$\frac{v^3}{v^4+1}dv = \frac{dx}{x}$$
$$\frac{1}{4}\ln\left(v^4+1\right) = \ln|x| + C$$
$$\ln\sqrt[4]{v^4+1} - \ln|x| = C$$
$$e^{\ln\sqrt[4]{v^4+1}-\ln|x|} = e^C$$

$$\frac{e^{\ln\sqrt[4]{v^4+1}}}{e^{\ln|x|}} = e^C$$

$$\frac{\sqrt[4]{v^4+1}}{|x|} = e^C$$

$$\frac{v^4+1}{x^4} = e^{4C}$$

$$v^4+1 = kx^4, \qquad \left(k = e^{4C}\right)$$

But $v = y/x$, then

$$\left(\frac{y}{x}\right)^4 + 1 = kx^4$$

$$y^4 + x^4 = kx^8$$

Exercises

Verify that the differential equation is homogeneous and solve it

1. $y' = \dfrac{2xy}{x^2 - y^2}$

2. $y' = \dfrac{x^2 + y^2}{xy}$

3. $\dfrac{dy}{dx} = e^{\frac{y}{x}} + \dfrac{y}{x}$

4. $\dfrac{dy}{dx} = \dfrac{-2x+4y}{x+y}$

5. $\dfrac{dy}{dx} = \dfrac{-x+3y}{x+y}$

6. $\dfrac{dy}{dx} = \dfrac{2x+y}{x+2y}$

7. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

8. $\dfrac{dy}{dx} = \dfrac{x^2+2y^2}{2xy+y^2}$

9. $\dfrac{dy}{dx} = \dfrac{y^4+x^3y}{x^4}$

10. $\dfrac{dy}{dx} = \dfrac{y^2+xy+x^2}{x^2}$

Chapter 3

Ordinary Differential Equations II (Higher-Order)

1 Second Order Nonlinear Equations

In general, second-order nonlinear differential equations are hard to solve. This section will present two substitutions which allow us to solve several important equations of the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

by solving two first order equations.

Case1: Dependent variable missing

Suppose the differential equation involves only the independent variable *x* and derivatives of the dependent variable *y*:

$$\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right)$$

then let $v = \frac{dy}{dx}$ and the equation becomes a first order equation $\frac{dv}{dx} = f(x, v)$ in *v* which can be solved by using previous techniques.

Example 1

Solve the initial value problem $y'' = 2x(y')^2, y(0) = 2, y'(0) = 1$

Solution

Note that the dependent variable *y* doesn't appear explicitly in the equation. Let $v = y'$. The initial condition $y'(0) = 1$ is then $v(0) = 1$ and the differential equation is

$$\frac{dv}{dx} = 2xv^2$$

which can be solved by separation of variable.

$$\int \frac{dv}{v^2} = \int 2x dx$$

or $-\frac{1}{v} = x^2 + C_1$

Applying the initial condition, we obtain $C_1 = -1$. Then

$$v = \frac{1}{1-x^2} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{1-x^2}$$

$$y = \int \frac{dy}{dx} dx = \int \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{|1+x|}{|1-x|} + C_2$$

where C_2 is a new arbitrary constant. By the initial condition $y(0)=2$ it implies that $C_2=2$ and hence the solution is

$$y=\frac{1}{2}\ln\frac{|1+x|}{|1-x|}+2$$

Example 2

Solve the differential equation $y''-\frac{y'}{x}=0,\ x>0$

Solution

Again the dependent variable y is missing from the equation. Let $v=y'$. The equation becomes

$$\frac{dv}{dx}-\frac{v}{x}=0$$

which can be considered a first order linear equation. The integrating factor is

$$\mu(x)=e^{-\int\frac{dx}{x}}=e^{-\ln x}=x^{-1}$$

Multiplying the equation by the integrating factor to get

$$\left(\frac{1}{x}v\right)'=0$$

Anti-differentiate and solve for v to obtain

$$v=C_1x \text{ or } y'=C_1x$$

Anti-differentiate again to find y:

$$y=C_1\frac{x^2}{2}+C_2$$

Case 2: Independent Variable Missing

In this case the equation is of the form $\frac{d^2y}{dx^2}=f\left(y,\frac{dy}{dx}\right)$; that is, it involves only the dependent variable and its derivatives.

Again let $v=dy/dx$ to get

$$\frac{dv}{dx}=f(y,v)$$

In order to reduce this to an equation in just y and v, observe that, by the chain rule,

$$\frac{dv}{dx}=\frac{dv}{dy}\frac{dy}{dx}=\frac{dv}{dy}v$$

Thus we can obtain

$$v\frac{dv}{dy}=f(y,v)$$

which is the first-order equation in v and y.

Example 3

Solve the initial-value problem $y''=(y')^3y, y(0)=1,\ y'(0)=-2$

Solution

Note that the independent variable is missing from the equation. Let

$$v = \frac{dy}{dx}, v\frac{dv}{dy} = \frac{d^2y}{dx^2}$$

When $x = 0$, then $y = 1, v = -2$ so $v(1) = -2$. The initial value problem becomes

$$v\frac{dv}{dy} = v^3 y, v(1) = -2$$

Proceed by separation of variables, assuming that $v \neq 0$.

$$\frac{1}{v^2}dv = ydy$$

and anti-differentiate, getting

$$-\frac{1}{v} = \frac{y^2}{2} + C_1$$

Applying the initial condition, we get $C_1 = 0$, then

$$v = -2/y^2$$

Thus

$$\frac{dy}{dx} = -\frac{2}{y^2}$$

This can be solved by separation of variables.

$$\int y^2 dy = -2\int dx$$

and anti-differentiation,

$$\frac{y^3}{3} = -2x + C_2$$

The initial condition $y(0) = 1$ implies $C_2 = \frac{1}{3}$ and the final result is

$$\frac{y^3}{3} = -2x + \frac{1}{3} \quad \text{or} \quad y = (1-6x)^{1/3}$$

Exercises

Solve the given second-order differential equation

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 2$
2. $y'' - y' = e^x$
3. $y'' + y = 0, y(0) = 1, y'(0) = 0$
4. $y'' + y = 0,\ y(0) = 0, y'(0) = 1$
5. $y'' = (y')^3 - (y')^2,\ y(0) = 3,\ y'(0) = 1$
6. $y'' = (y')^3 + y'$
7. $y'' = 2x(y')^2,\ y(0) = 0, y'(0) = -1$
8. $y'' = 2yy',\ y(0) = 0, y'(0) = -1$
9. $y^2 y'' = y'$
10. $-x + y'y'' = 0,\ y(1) = 0, y'(1) = 1$

2 Homogeneous Linear Differential Equation

2.1 Linear Independence

A **Linear differential equation** is one of the general forms

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = R(x)$$

and if $a_n(x) \neq 0$, it is said to be of **order** *n*. It is called *homogeneous* if $R(x)=0$ and *non-homogenous* if $R(x) \neq 0$. First we focus attention on the homogeneous case and next the non-homogeneous one.

Definition
The function $y_1(x), y_2(x), \ldots, y_n(x)$ are said to be **linear independent** if the equation
$C_1y_1 + C_2y_2 + \cdots + C_ny_n = 0$ for constants $C_1, \ldots, C_n$ has only the trivial solution $C_1 = C_2 = \cdots = C_n = 0$ for all *x* in the interval *I*. Otherwise they are said to be **linearly dependent**.

Example 1
The function $y_1 = \cos x$ and $y_2 = x$ are linearly independent for the only one way we can have $C_1 \cos x + C_2 x = 0$ for all *x* is for C_1 and C_2 both to be 0. However, the functions $y_1 = 1, y_2 = \sin^2 x$ and $y_3 = \cos 2x$ are linear dependent, because $C_1(1) + C_2(\sin^2 x) + C_3(\cos 2x) = 0$ for $C_1 = 1, C_2 = -2, C_3 = -1$, which is not the trivial solution.

Example 2
The function $y_1 = e^x, y_2 = 4e^x$ are linearly dependent on the interval $(-\infty, +\infty)$ since $-4y_1 + y_2 = -4e^x + 4e^x = 0$.

2.2 Wronskian

Suppose the coefficients $a_0, \ldots, a_n$ are continuous functions of *x* on the interval $a \le x \le b$ and $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

then the function $y_1, y_2, \ldots, y_n$ are linearly independent on $[a,b]$ if and only if the determinant

$$W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

for some *x* on $[a,b]$. The determinant is called the Wronskian function of the *n* functions on $[a,b]$.

Example 3
The functions $y_1 = e^{-x}, y_2 = xe^{-x}, y_3 = e^{3x}$ are solutions of a certain homogeneous linear differential equation with constant coefficients. Show that these solutions are linear independent.

Solution

$$W\left(e^{-x}, xe^{-x}, e^{3x}\right) = \begin{vmatrix} e^{-x} & xe^{-x} & e^{3x} \\ -e^{-x} & (1-x)e^{-x} & 3e^{3x} \\ e^{-x} & (x-2)e^{-x} & 9e^{3x} \end{vmatrix} = 16e^{x}$$

Hence the functions are linearly independent.

Example 4

The functions $y_1 = \sin 2x$ and $y_2 = \cos 2x$ are solutions of the second-order equation $y'' + 4y = 0$. Show that they form a linearly independent set of functions.

Theorem

If $a_0, \ldots, a_n$ are continuous functions of x if $a_n(x) \neq 0$ on the interval $[a,b]$, then the n^th order homogeneous linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

has n linearly independent solutions $y_1, \ldots, y_n$ on $[a,b]$ and, by the proper choice of constants $c_1, \ldots, c_n$ every solution of the equation can be expressed as

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

Example 5

(a) Show that $y = e^{2x}$ and $y = e^{-3x}$ are solutions of $y'' + y' - 6y = 0$

(b) Show that $y = 3e^{2x} + 5e^{-3x}$ is also a solution of this equation.

(c) Show that $y = xe^{2x}$ is not a solution.

3 Reduction of Order

Theorem

If y_1 is a nontrivial solution of the n^th order homogeneous linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

then the substitution $y_2 = y_1v$, followed by theq substitution $w = v'$, reduced the equation to (n-1)^th order equation.

Example 1

(a) Show that $y_1 = e^{-x}$ is a solution of $y'' + 3y' + 2y = 0$.

(b) Use the method of reduction of order to find a second linearly independent solution of this differential equation and write the general solution.

Solution

(a) Substituting $y_1 = e^{-x}, y_1' = -e^{-x}, y_1'' = e^{-x}$ into the given equation yields

$$e^{-x} + 3\left(-e^{-x}\right) + 2\left(e^{-x}\right) = 0$$

which shows that $y_1 = e^{-x}$ is a solution of the given equationl.

(b) Using the method of reduction of orderw, we let $y_2 = ve^{-x}$,which differentiated twicew, yields

$$y_2' = v'e^{-x} - ve^{-x}$$

$$y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$$

Substituting into the given differential equation, we get

$$\left(v''e^{-x} - 2v'e^{-x} + ve^{-x}\right) + 3\left(v'e^{-x} - ve^{-x}\right) + 2ve^{-x} = 0$$

Expanding and collecting terms yields

$$v''e^{-x} + v'e^{-x} = 0 \qquad \text{or} \qquad v'' + v' = 0$$

Letting $w = v'$, this becomes

$$w' + w = 0$$

Separating variables on this equation yields

$$\frac{dw}{w} = -dx$$

$$\ln|Cw| = -x$$

$$w = Ce^{-x}$$

Since $w = v'$, it follows by taking the antiderivative of e^{-x} that $v = ce^{-x}$

Ignoring the coefficient, a second solution is

$$y_2 = ve^{-x} = e^{-x}e^{-x} = e^{-2x}$$

The general solution of the given second order equation is then

$$y = c_1e^{-x} + c_2e^{-2x}$$

Exercises

Show that the given function is a solution of the diferential equation, use the method of reduction of oreder to find a second linearly independent solution, and write the general solution.

1. $2x^2y'' + xy' - y = 0; y_1 = x$

2. $y'' - 4y = 0; y_1 = e^{2x}$

3. $y'' - 4y = 0; y_1 = \cosh 2x$

4. $y'' - 9y = 0; y_1 = e^{3x}$

5. $y'' + y' - 6y = 0; y_1 = e^{-3x}$

6. $y'' + 4y = 0; y_1 = \sin 2x$

7. $xy'' + y' = 0; y_1 = 1$

8. $x^2y'' - 6y = 0; y_1 = 1/x^2$

9. $(1-x)y'' + xy' - y = 0; y_1 = e^x$

10. $x^2y'' - 3xy' + 4y = 0; y_1 = x^2$

4 Homogeneous Linear Equation with Constant Coefficients

It has the general form

$$b_ny^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1y' + b_0y = 0$$

where $b_0, b_1, \ldots, b_n$ are real constants. The euation

$$b_nr^n + b_{n-1}r^{n-1} + \cdots + b_1r + b_0 = 0$$

is called the *characteristic* or *auxiliary equation* associated with the given homogeneous linear equation with constant coefficients.

Example 1

(a) The auxiliary equation for $y' - 3y = 0$ is $r - 3 = 0$.

(b) The auxiliary equation for $y'' + 5y' - 7y = 0$ is $r^2 + 5r - 7 = 0$

(c) Equation such as $y'' + yy' = 0, y'' + y + x^2 = 0,$ or $x^2y'' + y' + xy = 0$ do not have auxiliary equations since the auxiliary equation concept applies only to linear homogeneous equation with constant coefficients.

4.1 Auxiliary Equation with Distinct Real Roots

If the auxiliary eqution for

$$b_ny^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1y' + b_0y = 0$$

has *n* distinct real roots $r_1, r_2, \ldots, r_n$ then *n* solutions $e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}$ are linearly independent and the general solution of the differential equation is given by

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx}$$

where $c_1,\ldots,c_n$ are arbitrary constants.

Example 2

Solve the differential equation $2\frac{d^3y}{dx^3}-9\frac{d^2y}{dx^2}-5\frac{dy}{dx}=0$

Solution

The auxiliary equation for the given differential equation is

$$2r^3-9r^2-5r=0$$

The roots of this equation are $r=0,-\frac{1}{2},5$; therefore the general solution is

$$y=c_1+c_2e^{-x/2}+c_3e^{5x}$$

Example 3

Solve the initial-value problem $y''+3y'+2y=0, y(0)=1, y'(0)=2$

Solution

The auxiliary equation in this case is

$$r^2+3r+2=0$$

whose roots are $r=-1,$ and -2. Therefore the general solution is

$$y=c_1e^{-x}+c_2e^{-2x}$$

To find c_1 and c_2, we use the condition $x=0, y=1$ in the general solution to obtain

$$1=c_1+c_2$$

Differentiating the general solution yields

$$y'=-c_1e^{-x}-2c_2e^{-2x}$$

Using $x=0, y'=2$ in this equation,

$$2=-c_1-2c_2$$

Solving the system for c_1 and c_2

$$\begin{cases}1=c_1+c_2\\ 2=-c_1-2c_2\end{cases}$$

we get $c_1=4, c_2=-3$.

Hence the solution is $y=4e^{-x}-3e^{-2x}$.

Exercises

Find the general solution

1. $y''-3y'+2y=0$

2. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

3. $\frac{d^2s}{dt^2}+\frac{ds}{dt}=0$

4. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=0$

5. $2y''-3y'=0$

6. $3y'-4y=0$

7. $\frac{d^2y}{dx^2}-4y=0$

8. $\frac{d^2i}{dt^2}-9i=0$

9. $y'''-16y'=0$

10. $y'''-4y'=0$

11. $y'''+9y''+8y'=0$

12. $3y'''+5y''-2y'=0$

Find the particular solution corresponding to the given conditions.

13. $\dfrac{d^2s}{dt^2}-4s=0; s(0)=0, \left.\dfrac{ds}{dt}\right|_{t=0}=2$

14. $y''-2y'-3y=0, y(0)=0, y'(0)=-4$

15. $y''-y=0, y(0)=1, y'(0)=1$

17. $y''+3y'=0, y(0)=2, y'(0)=6$

4.2 Auxiliary Equation with Repeated Real Roots

If the auxiliary equation for

$$b_ny^{(n)}+b_{n-1}y^{(n-1)}+\cdots+b_1y'+b_0y=0$$

has n repeated real roots $r_1=r_2=\cdots=r_n=r$, then the general solution is given by

$$y=\left(c_1+c_2x+c_3x^2+\cdots+c_nx^{n-1}\right)e^{rx}$$

where $c_1,\ldots,c_n$ are arbitrary constants.

Example 4

Solve the equation $y'''=0$.

Solution

The auxiliary equation is $r^3=0$ which gives roots m=0,0, and 0. Therefore the general solution is

$$y=c_1+c_2x+c_3x^2$$

Example 5

Solve the equation $y'''+4y''+4y'=0$

Solution

The auxiliary equation is

$$r^3+4r^2+4r=0$$

which has roots 0, -2, -2. Therefore the general solution is

$$y=c_1+c_2e^{-2x}+c_3xe^{-2x}$$

4.3 Auxiliary Equation with Complex Roots

If the auxiliary equation for

$$b_ny^{(n)}+b_{n-1}y^{(n-1)}+\cdots+b_1y'+b_0y=0$$

has the complex roots $r=a\pm ib$ then for each such pair of roots the general solution contains terms of the form

$$y=e^{ax}\left(c_1\cos bx+c_2\sin bx\right)$$

Example 6

Solve the equation $\dfrac{d^3s}{dt^3}+4\dfrac{ds}{dt}=0$

Solution

The auxiliary equation is $r^3+4r=0$, which has the roots $r_1=0, r_2=2i, r_3=-2i$. Therefor the general solution is

$$y=c_1+c_2\cos 2x+c_3\sin 2x$$

Example 7

Solve the equation $y^{(4)}+8y''+16y=0$

Solution

The auxiliary equation is $r^4+8r^2+16=0$ or $\left(r^2+4\right)^2=0$. The roots are $r_1=r_2=2i, r_3=r_4=-2i$. Therefore the general solution is

$$y=c_1\cos 2x+c_2\sin 2x+c_3x\cos 2x+c_4x\sin 2x$$

Exercises for 4.2 and 4.3

Find the general solution of the given differential equation

1. $y''+8y'+16y=0$
2. $4y''-4y'+y=0$
3. $y''+\dfrac{2}{3}y'+\dfrac{1}{9}y=0$
4. $y''+5y'+4y$
5. $y^{(5)}-y^{(3)}=0$
6. $\dfrac{d^4s}{dt^4}-\dfrac{d^2s}{dt^2}=0$
7. $y^{(4)}+18y'''+81y''=0$
8. $9y^{(4)}+6y'''+y''=0$
9. $4y'''-3y'+y=0$
10. $y'''-3y'-2y=0$

Solve the initial- value problem

11. $y''-8y'+16y=0, y(0)=0, y'(0)=1$
12. $y''-2y'+1=0, y(0)=1, y'(0)=2$
13. $y''-6y'+9y=0, y(0)=1, y'(0)=1$
14. $y'''+3y''=0, y(0)=3, y'(0)=0, y''(0)=9$
15. $y''+4y=0, y(0)=0, y'(0)=1$
16. $y''+y=0, y(0)=0, y'(0)=1$
17. $y''+4y'+5y=0, y(0)=1, y'(0)=0$
18. $y''-6y'+10=0, y(0)=2, y'(0)=1$

5 Non-homogeneous Linear Differential Equation with constant Coefficients

The n[th]-oder nonhomogeneous linear differential equation can be expressed in the form

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_0(x)y=f(x)$$

where $a(x)$ and $f(x)$ are not identically zero on some interval $a\le x\le b$. The function $f(x)$ is often called the **driving function** of the equation.

Any function y_p that is free of arbitrary constants and satisfies the equation is called a **particular solution** of the equation.

Example 1

(a) $y_p=5x$ is a particular solution of $y''+y'=5$

(b) $y_p=2e^{3x}$ is a particular solution of $y''-2y'+y=8e^{3x}$

Associated with the equation

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_0(x)y=f(x)$$

is the homogeneous linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

which is called the **corresponding homogeneous equation**.

Example 2

(a) The corresponding homogeneous equation for $y'' - 2y' - 3y = \sin x$ is $y'' - 2y' - 3y = 0$.

(b) The corresponding homogeneous equation for $y'' + y = 25$ is $y'' + y = 0$

The general solution of the corresponding homogeneous equation, denoted by y_c is called the **complementary solution** of the nonhomogeneous equation.

Let y_p be any particular solution of the n$^{\text{th}}$-order constant-coefficient linear differential equation

$$b_n y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1 y' + b_0 y = f(x) \qquad (\blacklozenge)$$

and let y_c be the general solution of the corresponding homogeneous equation

$$b_n y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1 y' + b_0 y = 0$$

Then the general solution of the equation (◆) is $y = y_c + y_p$.

5.1 Using the Method of Reduction of Order to Find a Particular Solution

Example 3

Solve the differential equation $y'' + 2y' + y = e^{3x}$ using the method of reduction of order to find a y_p.

Solution

The corresponding homogeneous equation is $y'' + 2y' + y = 0$. The auxiliary exquation is $r^2 + 2r + 1 = 0$, which has repeated roots -1,-1, yields the complementary solution

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

To find y_p by reduction of order, we let $y_p = vy_1$, where y_1 is any particular solution of the corresponding homogeneous equation. Here we choose $y_1 = e^{-x}$. Thus,

$$y_p = ve^{-x}$$

$$y_p' = v'e^{-x} - ve^{-x}$$

$$y_p'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$$

Substituting into the given equation, we get

$$\left(v''e^{-x} - 2v'e^{-x} + ve^{-x}\right) + 2\left(v'e^{-x} - ve^{-x}\right) + ve^{-x} = e^{3x}$$

This reduces to

$$v''e^{-x} = e^{3x} \text{ or } v'' = e^{4x}$$

whence

$$v = \frac{1}{16}e^{4x}$$

(Note: We omit the arbitrary constants because we're finding a particular solution)
Substitute the value of v in $y_p = ve^{-x}$, a particular solution of the given differential equation is

$$y_p = \frac{1}{16}e^{4x}e^{-x} = \frac{1}{16}e^{3x}$$

Finally, the general solution is

$$y = y_c + y_p = c_1e^{-x} + c_2xe^{-x} + \frac{1}{16}e^{3x}$$

Example 4
Solve the differential equation $y'' + 2y' = 3x$ using the method of reduction of order to find a y_p. Answer: $y = c_1 + c_2e^{-2x} + \frac{3}{4}x^2 - \frac{3}{4}x$.

Exercises
Determine the complementary Solution of the homogeneous equation

1. $y'' + 3y' + 2y = 12e^x$
2. $y'' + 2y' - 8y = 4$
3. $y'' + 6y' + 9y = 9x + 2$
4. $y'' - 4y' + 4y = 5x^2 + e^{-x}$
5. $y'' + 4y = e^{-x}$
6. $y'' + 16y = 2\sin 3x$
7. Verify that $y_p = 2e^x$ is a particular solution of $y'' + 3y' + 2y = 12e^x$ and then find the general solution.
8. Verify that $y_p = x - \frac{4}{9}$ is a particular solution of $y'' + 6y' + 9y = 9x + 2$ and then find the general solution.
9. Verify that $y_p = \frac{1}{5}e^{-x}$ is a particular solution of $y'' + 4y = e^{-x}$ and then find the general solution.
10. Verify that $y_p = \frac{2}{7}\sin 3x$ is a particular solution of $y'' + 16y = 2\sin 3x$ and then find the general solution.

Find the general solution of the differential equations. In determining y_p, use the method of reduction of order

11. $y'' - 4y = 3$
12. $y'' - y = x$
13. $y'' + 4y' + 4y = e^{-2x}$
14. $y'' + 3y' = e^x$
15. $y'' + 3y' + 2y = 25$
16. $y'' + 4y' = e^{2x}$
17. $y'' - y' = \sin x$
18. $y'' - y = e^{3x}$
19. $y'' - y' - 6y = 5$
20. $y'' + y' = x$

5.2 Method of undetermined coefficients

We use this method to find a particular solution y_p of the constant-coefficient nonhomogeneou linear equation

$$b_ny^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1y' + b_0y = f(x)$$

❶*If $f(x)$ is an m^{th}-degree polynomial $p(x)$, we assume*

$$y_p = x^k\left(A_0 + A_1x + \cdots + A_mx^m\right)$$

where k is the multiplicity of the root 0 of the auxiliary equation.

Example 5

Find the general solution of $y''-3y'=2x^2+1$.

Solution

The characteristic equation is $r^2-3r=0$ has the roots $r_1=0$ and $r_2=3$. Hence

$$y_c=c_1e^{0x}+c_2e^{3x}=c_1+c_2e^{3x}$$

Here $f(x)=2x^2+1$, 0 is a root of the multiplicity 1, so $k=1$.

Thus $y_p=x\left(A_0+A_1x+A_2x^2\right)=A_0x+A_1x^2+A_2x^3$

$$y_p'=A_0+2A_1x+3A_2x^2$$

$$y_p''=2A_1+6A_2x$$

Substituting these into the Eq. gives

$$2A_1+6A_2x-3\left(A_0+2A_1x+3A_2x^2\right)=2x^2+1$$

$$2A_1+6A_2x-3A_0-6A_1x-9A_2x^2=2x^2+1$$

Equating coefficients of like powers of *x*,

$$1:\quad 2A_1-3A_0=1$$

$$x:\quad 6A_2-6A_1=0$$

$$x^2:\quad -9A_2=2$$

We get $A_2=-\frac{2}{9}, A_1=-\frac{2}{9}, A_0=-\frac{13}{27}$. Thus the particular solution is

$$y_p=-\frac{13}{27}x-\frac{2}{9}x^2-\frac{2}{9}x^3$$

and the general solution is

$$y=y_p+y_c=-\frac{13}{27}x-\frac{2}{9}x^2-\frac{2}{9}x^3+c_1+c_2e^{3x}$$

❷*If $f(x)$ is of the form of $Ee^{\alpha x}$, then y_p has the form of $x^kAe^{\alpha x}$, where k is the multiplicity of the root α of the auxiliary equation.*

Example 6

Find the general solution of $y''-5y'-6y=4e^{2x}$

Solution

The related homogeneous equation is $y''-5y'-6y=0$. The characteristic equation is $r^2-5r-6=0$ and has roots $6,-1$. Thus, $y_c=c_1e^{6x}+c_2e^{-x}$. Here $\alpha=2$ so $k=0$ since 2 is *not a root* of characteristic equation.

Thus we assign $y_p=Ae^{2x}$. Substituting this into the equation gives,

$$\left(Ae^{2x}\right)''-5\left(Ae^{2x}\right)'-6\left(Ae^{2x}\right)=4e^{2x}$$

$$4Ae^{2x}-10Ae^{2x}-6Ae^{2x}=4e^{2x}$$

$$-12Ae^{2x}=4e^{2x}$$

So $A=-\frac{1}{3}$. Thus $y_p=-\frac{1}{3}e^{2x}$. Hence, the general solution is

$$y=y_p+y_c=-\frac{1}{3}e^{2x}+c_1e^{6x}+c_2e^{-x}$$

❸*Case3: If* $f(x)$ *has the form of* $p(x)e^{\alpha x}$ $p(x)$ *is an m^th^-degree polynomial, then* y_p *is assigned to be* $x^k\left(A_0 + A_1x + \cdots + A_mx^m\right)e^{\alpha x}$ *where* k *is the multiplicity of the root* α *of the auxiliary equation.*

Example 7

Write the form of y_p, given that $y'' - y' = x^3 + x + e^x - 2xe^x$

Solution

The characteristic equation is $r^2 - r = 0$ which has the roots $0, 1$. So $y_c = c_1 + c_2e^x$.
Here $f(x) = \left(x^3 + x\right) + \left(1 - 2x\right)e^x$. The first term is a third-degree polynomial. Since 0 is a root of multiplicity 1 of the characteristic equation, y_p must include a term of the form $x^k\left(A_0 + A_1x + A_2x^2 + A_3x^3\right)$ with $k = 1$.The second term is the form $p(x)e^{\alpha x}$ where $p(x) = 1 - 2x$ is the first-degree polynomial and $\alpha = 1$ which is also the root of the characteristic equation. So y_p must also include a term of the form $x^k\left(A_4 + A_5x\right)e^x$ with $k = 1$. So y_p has the form

$$y_p = x\left(A_0 + A_1x + A_2x^2 + A_3x^3\right) + x\left(A_4 + A_5x\right)e^x$$

Example 8

Give the form for y_p if $y'' - 2y' + y = 7xe^x$ is to be solved by the method of undetermined coefficients.

Solution

The characteristic equation is $r^2 - 2r + 1 = 0$ which has root 1 of multiplicity 2. Thus

$$y_c = c_1e^x + c_2xe^x.$$

Here $f(x)$ has the form $p(x)e^{\alpha x}$ where $p(x) = 7x$ is a first-degree polynomial and $\alpha = 1$. Since $\alpha = 1$ is a root of multiplicity 2, we obtain $k = 2$. So the form for y_p is

$$x^2\left(A_0 + A_1x\right)e^x$$

❹*Case 4: If* $f(x) = E_1\cos\beta x + E_2\sin\beta x$, *where at least one of the constants* E_1, E_2 *is nonzero, then* y_p *has the form* $x^k\left(A_0\cos\beta x + B_0\sin\beta x\right)$ *where* k *is the multiplicity of* βi *as a root of the auxiliary equation.*

Example 9

Write the form of y_p, if $y'' + 2y' + 2y = 3e^{-x} + 4\cos x$ is to be solved by the method of undetermined coefficient.

Solution

The characteristic equation is $r^2 + 2r + 2 = 0$ with the roots $r = -1 \pm i$.
Thus

$$y_h = c_1e^{-x}\cos x + c_2e^{-x}\sin x$$

Here $f(x) = 3e^{-x} + 4\cos x$. Consider the first term $3e^{-x}$. Since $\alpha = -1$ is not the root of the characteristic equation, by case 2, we have $k = 0$. So y_p includes a term of A_1e^{-x}.
Now consider the second term $4\cos x$. Since i is not a root of the characteristic equation, k=0. Hence y_p includes terms of the form $A_2\cos x + A_3\sin x$

Thus $y_p = A_1e^{-x} + A_2\cos x + A_3\sin x$, where A_1, A_2, A_3 are constants to be determined.

Example 10

Give the form for y_p if $y'' + 4y = \sin 2x$ is to be solved by the method of undetermined coefficients.

Solution

The characteristic equation is $r^2 + 4 = 0$ with the roots $\pm 2i$.

So $y_h = c_1\cos 2x + c_2\sin 2x$. $f(x) = \sin 2x$, that is $\beta = 2$. Since $2i$ is a roots of the characteristic equation of multiplicity 1, we have $k = 1$.

Hence

$$y_p = x\left(A_0\cos 2x + B_0\sin 2x\right)$$

❺*Case 5: If* $f(x) = p(x)\sin\beta x + q(x)\cos\beta x$, *where* $p(x)$ *is an* m^{th}*-degree polynomial in* x *and* $q(x)$ *is an* n^{th}*-degree polynomial in* x, *then*

$$y_p = x^k\left[\left(A_0 + A_1x + \cdots + A_sx^s\right)\cos\beta x + \left(B_0 + B_1x + \cdots + B_sx^s\right)\sin\beta x\right]$$

where k *is the multiplicity of* βi *as a root of the auxiliary polynomial and* s *is the larger of* m, n.

Example 11

Give the form for y_p if $y'' + 4y = x^2\cos 2x - x\sin 2x + \sin 2x$ is to be solved by the method of undetermined coefficients.

Solution

The characteristic equation is $r^2 + 4 = 0$ which has the roots $\pm 2i$.

So

$$y_h = c_1\cos 2x + c_2\sin 2x.$$

Here $f(x) = x^2\cos 2x - x\sin 2x + \sin 2x = x^2\cos 2x + (1 - x)\sin 2x$, that is, it has the form of

$$p(x)\cos\beta x + q(x)\sin\beta x$$

where $p(x) = x^2$ is a second-degree polynomial and $q(x) = 1 - x$ is a 1st degree polynomial, and $\beta = 2$. Since $2i$ is a root of the characteristic equation of multiplicity 1, we obtain $k = 1$. Hence

$$y_p = x\left[\left(A_1 + A_2x + A_3x^2\right)\cos 2x + \left(A_4 + A_5x + A_6x^2\right)\sin 2x\right]$$

❻*Case 6: If* $f(x) = E_1e^{\alpha x}\cos\beta x + E_2e^{\alpha x}\sin\beta x$, *where* E_1, E_2 *are constants at least one of which is nonzero, then*

$$y_p = x^k\left[A_0e^{\alpha x}\cos\beta x + B_0e^{\alpha x}\sin\beta x\right]$$

where k *is the multiplicity of* $\alpha + \beta i$ *as a root of the auxiliary polynomial.*

Example 12

Give the form for y_p if $y'' + 2y' + 2y = 5x^{-x}\cos x$ is to be solved by the method of undetermined coefficients.

Solution

The roots of the characteristic equation $r^2+2r+2=0$ are $-1\pm i$. So

$$y_h = c_1e^{-x}\cos x + c_2e^{-x}\sin x$$

Here $f(x)=5e^{-x}\cos x$ which is of the form $e^{\alpha x}\cos\beta x$, where $\alpha=-1,\beta=1$. Since $-1+i$ is a root of the characteristic equation of multiplicity 1, we obtain $k=1$. Thus

$$y_p = x\left(A_0e^{-x}\cos x + B_0e^{-x}\sin x\right).$$

We can summary the 6 cases as in the table below:

If $f(x)$ ***is of the form***	y_p ***then includes***	k ***is the multiplicity of the root***
1. $p(x)$**,** an m[th]-degree polynomial	$x^k\left(A_0+A_1x+\cdots+A_mx^m\right)$	0
2. $Ee^{\alpha x}$	$x^kAe^{\alpha x}$	α
3. $p(x)e^{\alpha x}$, $p(x)$ is an m^{th}-degree polynomial	$x^k\left(A_0+A_1x+\cdots+A_mx^m\right)e^{\alpha x}$	α
4. $E_1\cos\beta x+E_2\sin\beta x$	$x^k\left(A_0\cos\beta x+B_0\sin\beta x\right)$	βi
5. $p(x)\cos\beta x+q(x)\sin\beta x$ where $p(x)$ is an m^{th}-degree polynomial and $q(x)$ is n^{th}-degree polynomial	$x^k\left(A_0+A_1x+\cdots+A_sx^s\right)\cos\beta x+$ $x^k\left(B_0+B_1x+\cdots+B_sx^s\right)\sin\beta x$ *s* is larger of *m*, *n*	βi
6. $E_1e^{\alpha x}\cos\beta x+E_2e^{\alpha x}\sin\beta x$	$x^ke^{\alpha x}\left(A_0\cos\beta x+B_0\sin\beta x\right)$	$\alpha+\beta i$

Example 13

Give the form for y_p if $y''+2y'+2y=e^{-x}\cos 2x+e^{-x}\sin 2x+e^{-x}-3\cos x$ is to be solved by undetermined coefficients

Solution

The characteristic equation is $r^2+2r+2=0$ which has roots $-1\pm i$

Here $f(x)$ is the some of 3 groups of terms

$e^{-x}\cos 2x+e^{-x}\sin 2x$: since $-1+2i$ is not a root, we include $A_0e^{-x}\cos 2x+B_0e^{-x}\sin 2x$ (by case 6)

e^{-x} : Since -1 is not a root, we include A_2e^{-x}

$-3\cos x$: since i is not a root, we include $A_1\cos x+B_1\sin x$

Hence

$$y_p = A_0e^{-x}\cos 2x+B_0e^{-x}\sin 2x+A_1\cos x+B_1\sin x+A_2e^{-x}$$

Exercises for 5.2

1. Consider the differential equation $y'''+2y''+y'=f(x)$. Determine the y_p to be used if $f(x)$ equals each of the following:

(a) x
(b) $x+2$
(c) $\sin x+x$
(d) e^{-x}
(e) xe^{-x}
(f) $\sinh x$
(g) $\sinh x+\cosh x$
(h) $x\left(1+e^{-x}\right)$

Solve the following equations

2. $y'' - 4y' + 4y = e^x$
3. $y'' - 4y' + 4y = e^x + 1$
4. $y'' - 4y' + 4y = e^{2x}$
5. $y'' - 4y' + 4y = \sin x$
6. $y'' - 4y' + 4y = xe^{2x} + e^{2x}$
7. $y'' - 4y' + 4y = xe^{2x}$
8. $y'' + y = \sin 2x$
9. $y'' + 4y = \sin 2x$
10. $y'' + 4y' = \sin 2x$
11. $y'' - 2y' + 5y = \sin 2x$
12. $y'' - 2y' + 5y = e^x \cos 2x$
13. $y''' - 3y' - 2y = \sin 2x$
14. $y''' + y'' = 1$

5.3 Variations of Parameters

This method can be used to seek for the particular solution of nth-order equation $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = f(x)$ where $y_1, y_2, \ldots, y_n$ are n linearly independent solution of the homogeneous equation are known, we seek a solution of the form $y_p = v_1(x)y_1(x) + v_2(x)y_2(x) + \cdots + v_n(x)y_n(x)$

To find $v_i\ (i = 1, \ldots, n)$, first solve the following linear equations simultaneously for v_i':

$$\begin{cases} v_1'y_1 + v_2'y_2 + \cdots + v_n'y_n = 0 \\ v_1'y_1' + v_2'y_2' + \cdots + v_n'y_n' = 0 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ v_1'y_1^{(n-2)} + v_2'y_2^{(n-2)} + \quad + v_n'y_n^{(n-2)} = 0 \\ v_1'y_1^{(n-1)} + v_2'y_2^{(n-1)} + \quad + v_n'y_n^{(n-1)} = f(x)/a_n(x) \end{cases}$$

Then integrate each v_i' to obtain v_i, disregarding all constants of integration. This is permissible because we are seeking only *one* particular solution.

Example 14

1. For the special case $n = 1$, the system reduces to

$$v_1'y_1 = f(x)/a_1(x)$$

2. For the case $n=2$, it becomes

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = f(x)/a_2(x) \end{cases}$$

3. For the case $n=3$:

$$\begin{cases} v_1'y_1 + v_2'y_2 + v_3'y_3 = 0 \\ v_1'y_1' + v_2'y_2' + v_3'y_3' = 0 \\ v_1'y_1'' + v_2'y_2'' + v_3'y_3'' = f(x)/a_3(x) \end{cases}$$

Scope of the Method

The method of variation of parameters can be applied to *all* linear differential equation. It is therefore more powerful than the method of undetermined coefficients, which is restricted to linear differential equations with constant coefficients and particular forms of $f(x)$. Nonetheless, in those cases where both methods are applicable, the method of undetermined coefficients is usually the more efficient and, hence, preferable.

As a practical matter, the integration for v_i' may be impossible to perform. In such an event, other methods must be employed.

Example 15

Solve $y'' - 2y' + y = \dfrac{e^x}{x}$

Solution

Here n=2 and $y_c = c_1e^x + c_2xe^x$; and hence $y_p = v_1e^x + v_2xe^x$

Since $y_1 = e^x, y_2 = xe^x$ and $f(x) = e^x/x$, it follows that

$$\begin{cases} v_1'e^x + v_2'xe^x = 0 \\ v_1'e^x + v_1'\left(e^x + xe^x\right) = \dfrac{e^x}{x} \end{cases}$$

By solving the system, we obtain $v_1' = -1$ and $v_2' = 1/x$. Thus,

$$v_1 = \int v_1'dx = -\int dx = -x \qquad v_2 = \int v_2'dx = \int \frac{dx}{x} = \ln|x|$$

Hence is $y_p = -xe^x + xe^x \ln|x|$. The general solution is therefore

$$y = y_c + y_p = c_1e^x + c_2xe^x - xe^x + xe^x \ln|x|$$
$$= c_1e^x + c_3xe^x + xe^x \ln|x|, \; (c_3 = c_2 - 1)$$

Example 16

Solve $y''' - y'' = e^x$ using the method of variation of parameters.

Solution

The characteristic equation is $r^3 - r^2 = 0$ with roots 0, 0 and 1. Hence homogeneous solution is $y_c = c_1 + c_2x + c_3e^x$, implying that

$$y_p = v_1y_1 + v_2y_2 + v_3y_3$$

So we have, here,

$$y_1 = 1, y_2 = x, y_3 = e^x \text{ and } f(x) = e^x$$

Thus we must solve the system below for v_1', v_2', v_3'

$$\begin{aligned} v_1'\cdot 1 + v_2'x + v_3'e^x &= 0 \\ v_1'\cdot 0 + v_2'\cdot 1 + v_3'e^x &= 0 \\ v_1'\cdot 0 + v_2'\cdot 0 + v_3'e^x &= e^x \end{aligned}$$

From the system, we obtain

$$v_3' = 1, v_2' = -e^x, v_1' = xe^x - e^x$$

It implies that

$$v_3 = x, v_2 = -e^x, v_1 = xe^x - 2e^x$$

Therefore

$$y_p = v_1y_1 + v_2y_2 + v_3y_3 = \left(xe^x - 2e^x\right)1 + \left(-e^x\right)x + x\left(e^x\right) = xe^x - 2e^x$$

The general solution is

$$y = y_p + y_c = xe^x - 2e^x + c_1 + c_2 x + c_3 e^x$$
$$= c_1 + c_2 x + xe^x + c_4 e^x, \quad (c_4 = c_3 - 2)$$

Exercises for 5.3

Solve the following equation using the method of variation of parameters

1. $y'' - y' - 2y = e^{2x}$
2. $y'' + 2y' + y = \dfrac{e^{-x}}{x}$
3. $y'' + 4y = \tan 2x$
4. $y'' + 4y = \tan^2 2x$
5. $y'' + 4y = \sin^2 2x$
6. $y'' - 3y' + 2y = \cos\left(e^{-x}\right)$
7. $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$
8. $y'' + 6y' + 9y = \dfrac{e^{-3x}}{x^2 + 1}$
9. $y'' + 2y' + y = e^{-x} \ln x$
10. $y'' + 2y' + y = \dfrac{e^{-x}}{x^3}$
11. Verify that x and $1/x$ are solutions to the differential equation $x^2 y'' + xy' - y = 0$ on $(0, \infty)$. Solve the equation $x^2 y'' + xy' - y = x^2 \ln x$
12. Use exercise 11 to solve the equation $x^2 y'' + xy' - y = x^2$
13. Verify that $y_1 = x$ and $y_2 = x \ln x$ are solution of the corresponding homogeneous equation of $x^2 y'' - xy' + y = \dfrac{1}{x}$ and then solve the differential equation.
14. Verify that x and e^x are solution to $(1 - x) y'' + xy' - y = 0$ on $(1, \infty)$. Solve the equation $(1 - x) y'' + xy' - y = (x - 1)^2 e^{-x}$.

Chapter 5

Linear System of ODEs

In some situation, we are required to find the function $y_1 = y_1(x), y_2 = y_2(x), \ldots, y_n = y_n(x)$ that satisfy a system of differential equations containing the variable *x*, the unknown functions $y_1, y_2, \ldots, y_n$ and their derivatives.

Consider the system of first order differential equations

$$\begin{cases} \dfrac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n) \\ \dfrac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n) \end{cases} \qquad (1)$$

where $y_1, y_2, \ldots, y_n$ are unknown functions and *x* is a variable. Such a system, to be solved by first derivative, is called **a normal system.**
Solving the system is to determine the function $y_1, y_2, \ldots, y_n$ satisfying (1) and the initial conditions

$$y_1(x_0) = y_{10}, y_2(x_0) = y_{20}, \ldots, y_n(x_0) = y_{n0} \qquad (2)$$

if there exist.

Now let solve the system (1).

Differentiate the first equation of the system (1) with respect to *x* we obtain

$$\frac{d^2 y_1}{dx^2} = \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y_1}\frac{dy_1}{dx} + \cdots + \frac{\partial f_1}{\partial y_n}\frac{dy_n}{dx}$$

Replacing derivatives $\dfrac{dy_1}{dx}, \dfrac{dy_2}{dx}, \cdots, \dfrac{dy_n}{dx}$ by the expressions $f_1, f_2, \ldots, f_n$ from (1), we obtain the equation

$$\frac{d^2 y_1}{dx^2} = F_2(x, y_1, y_2, \ldots, y_n)$$

Differentiating the equation obtained and following the above procedure to get

$$\frac{d^3 y_1}{dx^3} = F_3(x, y_1, y_2, \ldots, y_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d^n y_1}{dx^n} = F_n(x, y_1, y_2, \ldots, y_n)$$

Hence we obtain the following system

$$\begin{cases} \dfrac{dy_1}{dx} = f_1\left(x, y_1, y_2, \dots, y_n\right) \\ \dfrac{d^2 y_1}{dx^2} = F_2\left(x, y_1, y_2, \dots, y_n\right) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d^n y_1}{dx^n} = F_n\left(x, y_1, y_2, \dots, y_n\right) \end{cases} \qquad (3)$$

Suppose we can obtain $y_2, y_3, \dots, y_n$ in functions of x, y_1, and the derivative $\dfrac{dy_1}{dx}, \dfrac{d^2 y_1}{dx^2}, \dots, \dfrac{d^{n-1} y_1}{dx^{n-1}}$ as follows

$$\begin{cases} y_2 = \varphi_2\left(x, y_1, y_1', \dots, y_1^{(n-1)}\right) \\ y_3 = \varphi_3\left(x, y_1, y_1', \dots, y_1^{(n-1)}\right) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_n = \varphi_n\left(x, y_1, y_1', \dots, y_1^{(n-1)}\right) \end{cases} \qquad (4)$$

Substituting these expressions in the last equation in (3) we obtain

$$\frac{d^n y_1}{dx^n} = \phi\left(x, y_1, y_1', \dots, y_1^{(n-1)}\right) \qquad (5)$$

We can find y_1 by solving (5)

$$y_1 = \psi_1\left(x_1, C_1, C_2, \dots, C_n\right) \qquad (6)$$

Differentiating this expression $(n-1)$ times with respect to *x*, we will find

$$\frac{dy_1}{dx}, \frac{d^2 y_1}{dx^2}, \dots, \frac{d^{n-1} y_1}{dx^{n-1}}$$

as function of $x, C_1, C_2, \dots, C_n$.

By substituting these functions into (4), we can determine $y_2, y_3, \dots, y_n$

$$\begin{aligned} &y_2 = \psi_2\left(x, C_1, C_2, \dots C_n\right) \\ &\dots\dots\dots\dots\dots\dots\dots \qquad (7) \\ &y_n = \psi_n\left(x, C_1, C_2, \dots, C_n\right) \end{aligned}$$

Example: Solve the system

$$\begin{cases} \dfrac{dy}{dx} = y + z + x \quad (a) \\ \dfrac{dz}{dx} = -4y - 3z + 2x \quad (b) \end{cases}$$

with the initial condition $y(0) = 1$ and $z(0) = 0$

Solution:
First differentiate the first equation with respect to *x* to obtain

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}+\frac{dz}{dx}+1$$

Substitute $\frac{dy}{dx}$ and $\frac{dz}{dx}$ from (a) and (b) into this above expression we obtain

$$\frac{d^2y}{dx^2}=\left(y+z+x\right)+\left(-4y-3z+2x\right)+1$$

$$\frac{d^2y}{dx^2}=-3y-2z+3x+1 \qquad \text{(c)}$$

From equation (a), we have $z=\frac{dy}{dx}-y-x$ (d)

The substitution of this expression into (c) gives

$$\frac{d^2y}{dx^2}=-3y-2\left(\frac{dy}{dx}-y-x\right)+3x+1$$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=5x+1 \qquad \text{(e)}$$

We find the general solution of (e) as

$$y=\left(c_1+C_2x\right)e^{-x}+5x-9 \qquad \text{(f)}$$

and we can also find

$$z=\left(C_2-2C_1-2C_2x\right)e^{-x}-6x+14 \qquad \text{(g)}$$

Now, by applying initial conditions $y\left(0\right)=1$ and $z\left(0\right)=0$, we will find C_1 and C_2.

Since $y\left(0\right)=1$, then from (f) we obtain $1=C_1-9$, implying that $C_1=10$ and $z\left(0\right)=0$, then from (g) we obtain $0=C_2-2C_1+14$, $C_2=6$

Hence, the solution is given by

$$y=\left(10+6x\right)e^{-x}+5x-9,\ z=\left(-14-12x\right)e^{-x}-6x+14$$

Example: Solve the system

$$\begin{cases}\dfrac{dx}{dt}=y+z\\ \dfrac{dy}{dt}=x+z\\ \dfrac{dz}{dt}=x+y\end{cases}$$

Solution:

Differentiating the first equation give

$$\frac{d^2x}{dt^2}=\frac{dy}{dt}+\frac{dz}{dt}$$

Then we obtain

$$\frac{d^2x}{dt^2}=x+z+x+y$$

or

$$\frac{d^2x}{dt^2}=2x+y+z$$

From the equation $\frac{dx}{dt} = y + z$, we can obtain $z = \frac{dx}{dt} - y$, then

$$\frac{d^2x}{dt^2} = 2x + y + \frac{dx}{dt} - y$$

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

This equation gives the general solution $x = C_1 e^{-t} + C_2 e^{2t}$. From this, we get

$$\frac{dx}{dt} = -C_1 e^{-t} + 2C_2 e^{2t}$$

From the third equation we have $y = \frac{dx}{dt} - x = -C_1 e^{-t} + 2C_2 e^{2t} - z$

Substituting *x* and *y* into the third equation we get

$$\frac{dz}{dt} + z = 3C_2 e^{2t}$$

which has the solution

$$z = C_3 e^{-t} + C_2 e^{2t}$$

Thus, $y = -C_1 e^{-t} + 2C_2 e^{2t} - \left(C_3 e^{-t} + C_2 e^{2t}\right) = -\left(C_1 + C_3\right) e^{-t} + C_2 e^{2t}$

Therefore, the general solution is give as

$$\begin{cases} x = C_1 e^{-t} + C_2 e^{2t} \\ y = -\left(C_1 + C_3\right) e^{-t} + C_2 e^{2t} \\ z = C_3 e^{-t} + C_2 e^{2t} \end{cases}$$

Exercises

Solve the following systems

1. $y_1' - y_2 = x^2$, $y_2' + 4y_2 = x$

2. $y_1' = y_2$, $y_2' = y_1$

3. $y_1' = 3y_1 + 2y_2$, $y_2' = y_1 - 5y_2$

4. $y_1' = 4y_1 + 3y_2$, $y_2' = y_1$

5. $y_1' = y_1 + y_2$, $y_2' = 3y_1 - y_2$

6. $y_1' = 5y_1 - 4y_2$, $y_2' = y_1 + 2y_2$

7. $y_1' + y_2' = 3$, $y_1' - y_2' = x$

8. $y_1' = y_2 + 1$, $y_2' = y_1 - 1$

9. $y_1' - y_2 = x^2$, $y_2' + 2y_1 = x$

10. $y_1' + y_2' = 1$, $y_1' + y_1 + y_2' - y_2 = 0$

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