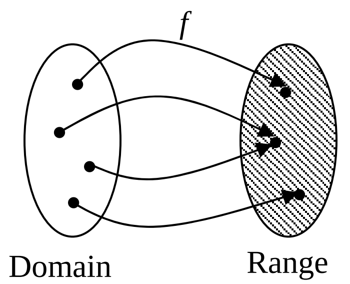
Functions, Limit, And Continuity

1. Definition of a function

A function is a rule of correspondence that associates with each object x in one set called the **domain,** a single value $f(x)$ from a second set. The set of all values so obtained is called the **range** of the function.



For a real function f we can define as follow

$$f:\mathbb{R}\rightarrow\mathbb{R}$$
$$x\mapsto y=f(x)$$

x can be called the **independent variable** and y the **dependent variable**. The domain of the function f, commonly denoted by D_f is defined by

$$D_f=\{\forall x\in\mathbb{R},\exists y\in\mathbb{R}\text{ such that } y=f(x)\}$$

Example:

1. $f(x)=\dfrac{1}{x}$ is defined for $x\neq 0$. Hence, $D_f=\mathbb{R}-\{0\}$
2. $f(x)=\sqrt{x^2-1}$ is defined for $x\geq 1$ and $x\leq -1$. Hence, $D_f=(-\infty,-1]\cup[1,+\infty)$
3. $f(x)=\sqrt{1-x^2}$ is defined for $-1\leq x\leq 1$

2. Composition of Functions

$$x\xrightarrow{f}f(x)\xrightarrow{g}g(f(x)),\qquad x\xrightarrow{g\circ f}g(f(x))$$

If f works on x to produce $f(x)$ and g works on $f(x)$ to produce $g(f(x))$, we say that we have composed g with f. The resulting function, called the **composition** of g with f, is denoted by $g\circ f$. Thus, $(g\circ f)(x)=g(f(x))$

Example: Given the function $f(x)=(x-3)/2$, $g(x)=\sqrt{x}$. Find $g\circ f$ and $f\circ g$

Solution

$$(g \circ f)(x) = g(f(x)) = \sqrt{\frac{x-3}{2}}$$

$$(f \circ g)(x) = f(g(x)) = \frac{\sqrt{x}-3}{2}$$

Example: Write the function $p(x) = (x+5)^5$ as a composite function $g \circ f$

Solution:

The most obvious way to decompose p is to write $p(x) = g(f(x))$, where $g(x) = x^5$, and $f(x) = x+2$

3. Inverse Functions

Inverse Function

Let f be a function with domain D and range R. Then the function f^{-1} with domain R and range D is the inverse of f if

$f^{-1}(f(x)) = x$ for all x in D

and

$f(f^{-1}(y)) = y$ for all y in R.

Example: Let $f(x) = 2x-3$. Find f^{-1} if it exists.

Solution:

To find f^{-1}, let $y = f(x)$, then interchange the x and y variables, and finally solve for y.

$y = 2x-3$, then $x = 2y-3$, implying $y = \frac{1}{2}(x+3)$, hence $f^{-1} = \frac{1}{2}(x+3)$

Criteria For Existence of An Inverse f^{-1}

A function f will have an inverse f^{-1} on the interval I when there is exactly one number in the domain associated with each number in the range. That is, f^{-1} exists if $f(x_1)$ and $f(x_2)$ are equal only when $x_1 = x_2$. A function with this property is said to be **one-to-one** function.

Horizontal Line Test

A function f has an inverse iff no horizontal line intersects the graph of $y = f(x)$ at more than one point.

A function is called to be strictly monotonic on the interval I if it is strictly increasing or strictly decreasing on that interval.

Strictly increasing on I: For $x_1, x_2 \in I$ such that $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Strictly decreasing on I: For $x_1, x_2 \in I$ such that $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Theorem

Let *f* be a function that is strictly monotonic on an interval *I*. Then f^{-1} exists and is monotonic on *I*.

Graph of f^{-1}

If f^{-1} exists, its graph may be obtained by reflecting the graph of *f* in the line $y = x$.

4. Inverse Trigonometric Functions

Inverse Sine Function

$$y = \sin^{-1} x \Leftrightarrow x = \sin y \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

The function $\sin^{-1} x$ is sometimes written as $\arcsin x$.

Inverse Tangent Function

$$y = \tan^{-1} x \Leftrightarrow x = \tan x \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

The function $\tan^{-1} x$ is sometimes written as $\arctan x$

Definition of Inverse Trigonometric Function

Inverse Function	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < +\infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \csc^{-1} x$	$x \ge 1 \text{ or } x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
$y = \sec^{-1} x$	$x \ge 1 \text{ or } x \le -1$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$
$y = \cot^{-1} x$	$-\infty < x < +\infty$	$0 < y < \pi$

Example: Evaluate the given function

a. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **b.** $\cos^{-1} 0$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Solution:

a. $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ **b.** $\cos^{-1} 0 = \frac{\pi}{2}$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Inverse Trigonometric Identities

Inversion Formulas

$\sin\left(\sin^{-1} x\right) = x$ for $-1 \le x \le 1$

$\sin^{-1}\left(\sin y\right) = y$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\tan\left(\tan^{-1} x\right) = x$ for all x

$\tan^{-1}\left(\tan y\right) = y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Example: Evaluate the given functions **a.** $\sin\left(\sin^{-1} 0.5\right)$ **b.** $\sin^{-1}\left(\sin 0.5\right)$

Solution:

a. $\sin\left(\sin^{-1} 0.5\right) = 0.5$ because $-1 \le 0.5 \le 1$

b. $\sin^{-1}\left(\sin 0.5\right) = 0.5$, because $-\frac{\pi}{2} \le 0.5 \le \frac{\pi}{2}$

Example: For $-1 \le x \le 1$, show that **a.** $\sin^{-1}\left(-x\right) = -\sin^{-1} x$ **b.** $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$

Some other Identities

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$

5. Hyperbolic Functions and Their Inverses

5.1 Definition

The hyperbolic sine and hyperbolic cosine function, denoted respectively by sinh and cosh, are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

The other hyperbolic function, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant and hyperbolic cosecant are defined in terms of sinh and cosh as follows

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\sec hx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad \csc hx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

5.2 Hyperbolic Identities

1/. $\cosh^2 x - \sinh^2 x = 1$

2/. $1 - \tanh^2 x = \sec h^2 x$

3/. $\coth^2 x - 1 = \csc h^2 x$

4a/. $\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$

4b/. $\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$

5a/. $\cosh x+\sinh x=e^{x}$

5b/. $\cosh x-\sinh x=e^{-x}$

6a/. $\sinh 2x=2\sinh x\cosh x$

6b/. $\cosh 2x=\cosh^2 x+\sinh^2 x$

7a/. $\cosh 2x=2\sinh^2 x+1$

7b/. $\cosh 2x=2\cosh^2 x-1$

8a/. $\cosh(-x)=\cosh x$

8b/. $\sinh(-x)=-\sinh x$

9a/. $\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y$

9b/. $\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y$

5.3 Inverse Hyperbolic Functions

The Hyperbolic inverses that are important and to be studied here are the *inverse hyperbolic sine,* the *inverse hyperbolic cosine,* and *inverse hyperbolic tangent.* These functions are $y=\sinh^{-1}x$ (or $y=Arc\sinh x$), $y=\cosh^{-1}x$ (or $y=Arc\cosh x$) and $y=\tanh^{-1}x$ (or $y=Arc\tanh x$) are the inverses of $y=\sinh x$, $y=\cosh x$ and $y=\tanh x$ respectively.

Theorem

i/. $\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$ (for any real number)

ii/. $\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$ ($x\geq 1$)

iii/. $\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ ($-1<x<1$)

6 Limits

Definition:

To say that $\lim_{x\to c}f(x)=L$ means that for each given $\varepsilon>0$ (no matter how small) there is a corresponding $\delta>0$ such that $|f(x)-L|<\varepsilon$ provided that $0<|x-c|<\delta$; that is $0<|x-c|<\delta\Rightarrow|f(x)-L|<\varepsilon$.

Example: 1/.Prove that $\lim_{x\to 4}(3x-7)=5$ 2/. $\lim_{x\to 2}\frac{2x^2-3x-2}{x-2}=5$

Right-hand Limit and Left-Hand Limit

By $\lim_{x\to a^-}f(x)=A$ we mean that *f* is defined in some open interval (c,a) and $f(x)$ aproaches *A* as *x* approaches *a* through values less than *a*, that is, as *x* approaches *a* from the left. Similarly, $\lim\limits_{x\to a^+} f(x) = A$ means that *f* is defined in some open interval (a,d) and $f(x)$ approaches *A* as *x* approaches *a* from the right.

If *f* is defined in an interval to the left of *a* and in an interval to the right of *a*, then $\lim\limits_{x\to a} f(x) = A$ iff $\lim\limits_{x\to a^-} f(x) = A$ and $\lim\limits_{x\to a^+} f(x) = A$

Limit Theorems

Let *n* be a positive integer, *k* be a constant, and *f* and *g* be functions that have limits at *c*. Then

1/. $\lim\limits_{x\to c} k = k$

2/. $\lim\limits_{x\to a} x = a$

3/. $\lim\limits_{x\to c}\left[kf(x)\right] = k\lim\limits_{x\to c} f(x)$

4/. $\lim\limits_{x\to c}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to c}\left[f(x)\right]\pm\lim\limits_{x\to c}\left[g(x)\right]$

5/. $\lim\limits_{x\to c}\left[f(x)g(x)\right] = \left[\lim\limits_{x\to c} f(x)\right]\left[\lim\limits_{x\to c} g(x)\right]$

6/. $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}, \lim\limits_{x\to c} g(x)\neq 0$

7/. $\lim\limits_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to c} f(x)}$ if defined.

7. Continuity of Functions

Continuity at a Point

Let *f* be defined on an open interval containing *c*. We say that *f* is continuous at *c* if $\lim\limits_{x\to c} f(x) = f(c)$.

Example: $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, x\neq 0 \\ 3 \qquad x = 0 \end{cases}$

At the point $x = 0$, f is defined and $f(0) = 3$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} = \dfrac{\sin 3x}{x} = 3$

We see that $\lim\limits_{x\to 0} f(x) = f(0)$. Thus f is continuous at the point $x = 0$

Example: Show that f is discontinuous at the point $x = 1$

$$f(x) = \begin{cases} -2x+4, & x>1 \\ x+1, & x<1 \\ -1, & x=1 \end{cases}$$

At the point $x = 1$ the function is defined, that is $f(1) = -1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(-2x+4) = 2$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+1) = 2$$

We see that $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$. Then $\lim_{x\to 1} f(x) = 2 \neq f(1)$

Hence f is discontinuous at the point $x=1$

Definition Continuity on an Interval

The function *f* is **right continuous** at *a* if $\lim_{x\to a^+} f(x) = f(a)$ and **left continuous** at *b* if $\lim_{x\to b^-} f(x) = f(b)$.

We say *f* is **continuous on an open interval** if it is continuous at each point of that interval. It is **continuous on the closed interval** $[a,b]$ if it is continuous on (a,b), right continuous at *a*, and left continuous at *b*.

***Example*:** Show that $f(x)=\sqrt{9-x^2}$ is continuous on the closed interval $[-3,3]$

Solution: We see that the domain of f is the interval $[-3,3]$. For *c* in the interval $(-3,3)$ we have

$$\lim_{x\to c} f(x) = \lim_{x\to c}\sqrt{9-x^2} = \sqrt{9-c^2} = f(c)$$

So f is continuous on $(-3,3)$. Also

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\sqrt{9-x^2} = 0 = f(3)$$

and

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+}\sqrt{9-x^2} = 0 = f(-3)$$

So f is continuous on $[-3,3]$.

Exercises

1 Given $\varphi(x)=\dfrac{x-1}{3x+5}$, determine $\varphi\left(\dfrac{1}{x}\right)$.

2 If $f(\alpha)=\tan(\alpha)$, verify that $f(2\alpha)=\dfrac{2f(\alpha)}{1-\left[f(\alpha)\right]^2}$.

3 Given $f(x)=\ln x$ and $\varphi(x)=x^3$, determine $(f\circ\varphi)(2)$, $(f\circ\varphi)(a)$ and $(\varphi\circ f)(a)$.

4 Find the domain of the following functions

a. $y=\sqrt{3-x^2}$ b. $f(x)=\sqrt{3+x}+\sqrt[4]{7-x}$ c. $y=\sqrt{\ln x+1}$

d. $y=\ln(\ln x)$ e. $y=\arcsin(3x-5)$

f. $y=\ln\left(x^2-3x+2\right)+\sqrt{-x^2+4x+5}$ g. $y=\dfrac{\sin x}{\sqrt{x^2-4}}$

5 If $f(x)=2^x$, show that a. $f(x+3)-f(x-1)=\frac{15}{2}f(x)$ b. $\frac{f(3+x)}{f(-1+x)}=f(4)$

6 If $f(x)=\frac{x-1}{x+1}$, show that $f\left(\frac{1}{x}\right)=-f(x)$ and $f\left(-\frac{1}{x}\right)=-\frac{1}{f(x)}$

7 If $f(x)=\frac{1}{x}$, then show that $f(a)-f(b)=f\left(\frac{ab}{b-a}\right)$

8 Compute $\frac{f(a+h)-f(a)}{h}$ in the following cases:

a. $f(x)=\frac{1}{x-2}$ when $a\neq 2$ and $a+h\neq 2$

b. $f(x)=\sqrt{x-4}$ when $a\geq 4$ and $a+h\geq 4$

c. $f(x)=\frac{x}{x+1}$ when $a\neq -1$ and $a+h\neq$ -1

9 Prove that

a. $\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right),\forall x\in\mathbb{R}$

b. $\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),(-1<x<1)$

10 Prove that

a. $\sin\left(\cos^{-1}x\right)=\sqrt{1-x^2}$ b. $\cos\left(\sin^{-1}x\right)=\sqrt{1-x^2}$

c. $\sec\left(\tan^{-1}x\right)=\sqrt{1+x^2}$ d. $\sin\left[2\cos^{-1}\left(\frac{2}{3}\right)\right]=\frac{4\sqrt{5}}{9}$

e. $\tan\left(\sin^{-1}x\right)=\frac{x}{\sqrt{1-x^2}}$ f. $\sin\left(\tan^{-1}x\right)=\frac{x}{\sqrt{1+x^2}}$

g. $\cos\left(2\sin^{-1}x\right)=1-2x^2$ h. $\tan\left(2\tan^{-1}x\right)=\frac{2x}{1-x^2}$

11 Prove that $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $-\frac{\pi}{2}<\tan^{-1}x+\tan^{-1}y<\frac{\pi}{2}$ and use the fact to prove that

a. $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$ b. $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$

12 Compute $\cos\left(\sin^{-1}\frac{1}{5}+2\cos^{-1}\frac{1}{5}\right),\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}\frac{1}{4}\right)$

13 Prove that $f(x)=x^2-3x+2$ is continuous at $x=4$

14 Prove that $f(x)=1/x$ is continuous at **a.** $x=2$

15 Investigate the continuity of each of the following functions at the indicated points:

a. $f(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ at the point $x=0$

b. $f(x)=x-|x|$ at the point $x=0$

c. $f(x)=\begin{cases}\dfrac{x^3-8}{x^2-4}, & x\neq 2\\ 3, & x=2\end{cases}$ at the point $x=2$

16 Find a value for the constant *k*, if possible, that will make the function continuous

a. $f(x)=\begin{cases}7x-2, & x\le 1\\ kx^2, & x>1\end{cases}$

b. $f(x)=\begin{cases}kx^2, & x\le 2\\ 2x+k, & x>2\end{cases}$

ans: **a.** 5 **b.** $4/3$

17 Find the points of discontinuity, if any, of the function $f(x)$ such that

$$f(x)=\begin{cases}x+1, & x\ge 2\\ 2x-1, & 1<x<2\\ x-1, & x\le 1\end{cases}$$

ans: discontinuous at $x=1$

18 If the function

$$f(x)=\begin{cases}\dfrac{x^2-16}{x-4}, & x\neq 4\\ c, & x=4\end{cases}$$

is continuous, what is the value of *c*?

ans: 8

19 For what value of *k* is the following a continuous function?

$$f(x)=\begin{cases}\dfrac{\sqrt{7x+2}-\sqrt{6x+4}}{x-2}, & \text{if } x\ge -\dfrac{7}{2} \text{ and } x\neq 2\\ k & \text{if } x=2\end{cases}$$

ans: $\dfrac{1}{8}$

20 Let

$$f(x)=\begin{cases}3x^2-1, & x<0\\ cx+d, & 0\le x\le 1\\ \sqrt{x+8}, & x>1\end{cases}$$

Determine *c* and *d* so that f is continuous (everywhere).

ans: $d=-1$, $c=4$

21 Determine if the following function is continuous at x=1.

$$f(x)=\begin{cases}3x-5, & x\neq 1\\ 2 & , x=1\end{cases}$$

22 Determine if the following function is continuous at x=-2.

$$f(x)=\begin{cases}x^2+2x, & x\leq -2\\ x^3-6x, & x>-2\end{cases}$$

23 Determine if the following function is continuous at x=0

$$f(x)=\begin{cases}\dfrac{x-6}{x-3}, & x<0\\ 2 & , x=0\\ \sqrt{4+x^2}, & x>0\end{cases}$$

24. Determine if the function $h(x)=\dfrac{x^2+1}{x^3+1}$ is continuous at x = -1.

25. For what values of x is the function $f(x)=\dfrac{x^2+3x+5}{x^2+3x-4}$ continuous?

Differentiation

1 Definition

A function f is said to be differentiable at x if and only if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists.

If this limit exists, it is called the derivative of f at x and is denoted by $f'(x)$. Hence,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Example $f(x)=x^2,\ f'(x)=?$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-x^2}{h}=\frac{x^2+2hx+h^2-x^2}{h}=2x+h$$

Then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}(2x+h)=2x$$

Example Find $f'(-2)$ if $f(x)=1-x^2$

Solution

We can first find $f'(x)$ in general

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[1-(x+h)^2\right]-\left[1-x^2\right]}{h}$$

$$=\lim_{h\to 0}\frac{-2xh-h^2}{h}=\lim_{h\to 0}(-2x-h)=-2x$$

and then substitute -2 for x

$$f'(-2)=-2\cdot(-2)=4$$

We can also evaluate $f'(-2)$ more directly

$$f'(-2)=\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}=\lim_{h\to 0}\frac{\left[1-(-2+h)^2\right]-\left[1-(-2)^2\right]}{h}=\lim_{h\to 0}\frac{4h-h^2}{h}=4$$

Example Find $f'(0)$ if $f(x)=\begin{cases}3x^2+1, & x\le 0\\ x^3+1, & 0<x<1\end{cases}$

Example Find the derivative of $f(x)=\dfrac{x}{x-9}$

The process of finding a derivative is called ***differentiation***. In the case where the independent variable is x it is denoted by the symbol

$$\frac{d}{dx}\left[f(x)\right]$$

read the derivative of $f(x)$ with respect to x

$$\frac{d}{dx}\left[f(x)\right] = f'(x)$$

If the dependent variable $y = f(x)$, then we write $\frac{dy}{dx} = f'(x)$

2 Rules for Differentiating Functions

Assume that *u*, *v*, and *w* are differentiable functions of *x* and that *c* and *m* are constants

1 $\frac{d}{dx}(c) = 0$ (The derivative of a constant is zero)

2 $\frac{d}{dx}(cu) = c\frac{du}{dx}$

3 $\frac{d}{dx}(x^m) = mx^{m-1}$ (Power Rule)

4 $\frac{d}{dx}(u \pm v \pm w) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$ (sum/difference rule)

5 $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ (Product Rule)

6 $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (Quotient Rule)

7 $\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{-\frac{du}{dx}}{u^2}, u \neq 0$ (Reciprocal Rule)

3 The Chain Rule

If we know the derivatives of f and g, how can we use this information to find the derivative of the composition $f \circ g$?

The key to solving this problem is to introduce dependent variables

$$y = (f \circ g)(x) = f(g(x)) \text{ and } u = g(x)$$

So that $y = f(u)$. We use the unknown derivatives

$$\frac{dy}{du} = f'(u) \text{ and } \frac{du}{dx} = g'(x)$$

to find the unknown derivative

$$\frac{dy}{dx} = \frac{d}{dx}\left[f(g(x))\right]$$

Theorem (The Chain Rule)

If g is differentiable at the point *x* and f is differentiable at the point $g(x)$ then the composition $f \circ g$ is differentiable at the point *x*. Moreover, if

$$y = f(g(x)) \text{ and } u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example Find $\frac{dy}{dx}$ if $y = 4\cos(x^3)$

Solution

Let $u = x^3$ so that $y = 4\cos u$, then by chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left[4\cos u\right] \cdot \frac{d}{dx}\left[x^3\right] = \left(-4\sin u\right) \cdot \left(3x^2\right) = -12x^2 \sin x^3$$

In general, if $f\left(g\left(x\right)\right)$ is a composite function in which the inside function g and the outside function f are differentiable, then

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right] = f'\left(g\left(x\right)\right) \cdot g'\left(x\right)$$

Example Find the derivative of $\left(\frac{x+2}{x-3}\right)^3$

Solution

By using the chain rule, we obtain

$$\frac{d}{dx}\left(\frac{x+2}{x-3}\right)^3 = 3\left(\frac{x+2}{x-3}\right)^2 \cdot \frac{d}{dx}\left(\frac{x+2}{x-3}\right)$$

Let calculate $\frac{d}{dx}\left(\frac{x+2}{x-3}\right)$

$$\frac{d}{dx}\left(\frac{x+2}{x-3}\right) = \frac{\left(x-3\right)\frac{d}{dx}\left(x+2\right) - \left(x+2\right)\frac{d}{dx}\left(x-3\right)}{\left(x-3\right)^2} = \frac{\left(x-3\right)\cdot 1 - \left(x+2\right)\cdot 1}{\left(x-3\right)^2} = -\frac{5}{\left(x-3\right)^2}$$

Hence

$$\frac{d}{dx}\left(\frac{x+2}{x-3}\right)^3 = 3\left(\frac{x+2}{x-3}\right)^2 \cdot \frac{d}{dx}\left(\frac{x+2}{x-3}\right) = 3\left(\frac{x+2}{x-3}\right)^2 \cdot \left(-\frac{5}{\left(x-3\right)^2}\right) = -15\frac{\left(x+2\right)^2}{\left(x-3\right)^3}$$

4 Derivatives of Trigonometric and Hyperbolic Functions

1 $\frac{d}{dx}\left(\sin x\right) = \cos x$

2 $\frac{d}{dx}\left(\cos x\right) = -\sin x$

3 $\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

4 $\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$

5 $\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$

6 $\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$

7 $\frac{d}{dx}\left(\cosh x\right) = \sinh x$

8 $\frac{d}{dx}\left(\sinh x\right) = \cosh x$

N.B: $\tan x = \frac{\sin x}{\cos x}$ $\cot x = \frac{\cos x}{\sin x}$ $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$ [1]

Proof

Recall that $\lim_{h \to 0} \frac{\sinh}{h} = 1$ and $\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$

From the definition of a derivative,

[1] sec: secant and csc: cosecant

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right] = \lim_{h\to 0}\left[\cos x\left(\frac{\sin h}{h}\right)-\sin x\left(\frac{1-\cos h}{h}\right)\right]$$

Since $\lim_{h\to 0}(\sin x) = \sin x$ and $\lim_{h\to 0}(\cos x) = \cos x$,

$$\frac{d}{dx}[\sin x] = \cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)-\sin x\cdot\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right) = \cos x\cdot(1)-\sin x\cdot(0) = \cos x$$

Thus, we have shown that

$$\frac{d}{dx}[\sin x] = \cos x$$

The derivative of $\cos x$ is obtained similarly:

$$\frac{d}{dx}[\cos x] = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$
$$= \lim_{h\to 0}\left[\cos x\cdot\left(\frac{\cos h-1}{h}\right)-\sin x\cdot\left(\frac{\sin h}{h}\right)\right]$$
$$= -\cos x\cdot\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)-\sin x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$
$$= (-\cos x)(0)-(\sin x)(1) = -\sin x$$

Thus, we have shown that

$$\frac{d}{dx}[\cos x] = -\sin x$$

Example Find $f'(x)$ if $f(x) = x^2\tan x$

Solution

$$f'(x) = x^2\cdot\frac{d}{dx}[\tan x]+\tan x\cdot\frac{d}{dx}\left[x^2\right]$$
$$= x^2\sec^2 x+2x\tan x$$

Example Find dy/dx if $y = \frac{\sin x}{1+\cos x}$

Solution

$$\frac{dy}{dx} = \frac{(1+\cos x)\cdot\frac{d}{dx}[\sin x]-\sin x\cdot\frac{d}{dx}[1+\cos x]}{(1+\cos x)^2} = \frac{(1+\cos x)(\cos x)-(\sin x)(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2} = \frac{\cos x+1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

5 Derivatives of Functions not Represented Explicitly

5-1 Implicit differentiation

Consider the equation $xy = 1$. One way to obtain dy/dx is to write the equation as $y = \frac{1}{x}$ from which it follows that

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Another way is to differentiate both sides

$$\frac{d}{dx}(xy)=\frac{d}{dx}(1)$$

$$x\frac{d}{dx}(y)+y\frac{d}{dx}(x)=0$$

$$x\frac{dy}{dx}+y=0$$

$$\frac{dy}{dx}=-\frac{y}{x}$$

Since $y=\frac{1}{x}$, $y'=\frac{dy}{dx}=-\frac{1}{x^2}$

This second method of obtaining derivatives is called ***implicit differentiation.***

Example By implicit differentiation find dy/dx if $5y^2+\sin y=x^2$

Solution

Differentiating both sides with respect to *x* and treating and treating *y* as a differentiable function of *x*, we obtain.

$$\frac{d}{dx}\left(5y^2+\sin y\right)=\frac{d}{dx}\left(x^2\right)$$

$$5\frac{d}{dx}\left(y^2\right)+\frac{d}{dx}(\sin y)=2x$$

$$5\left(2y\frac{dy}{dx}\right)+(\cos y)\frac{dy}{dx}=2x$$

$$10y\frac{dy}{dx}+\cos y\frac{dy}{dx}=2x$$

$$\frac{dy}{dx}=\frac{2x}{10y+\cos y}$$

Example Find $\frac{dy}{dx}$ if $7x^4+x^3y+x=4$

Example Find $\frac{d^2y}{dx^2}$ if $4x^2-2y^2=9$

5-2 Derivative of the Inverse Functions

Let $y=f(x)$ be a function whose inverse is $x=f^{-1}(y)$. Then $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$

Example Find the derivative of $y=\arcsin x$.

Solution

We have $y=\arcsin x \Leftrightarrow x=\sin y$ and hence $\frac{dx}{dy}=\frac{d}{dy}(\sin y)=\cos y$. Then

$$\frac{dy}{dx}=\frac{d}{dx}(\arcsin x)=\frac{1}{\frac{dx}{dy}}=\frac{1}{\cos y}=\frac{1}{\cos(\arcsin x)}=\frac{1}{\sqrt{1-x^2}}$$

Example Find the derivative of $y=\arccos x$ and $y=\arctan x$.

5-3 Derivatives of functions Represented Parametrically

If a function *y* is related to a variable *x* by means of a parameter *t*

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

or, in other notation,

$$y'_x = \frac{y'_t}{x'_t}$$

Example Find $\frac{dy}{dx}$ if $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$

Solution

We find $\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = a\cos t$. Hence $\frac{dy}{dx} = -\frac{a\cos t}{a\sin t} = -\cot t$.

Example Find $\frac{dy}{dx}$ if $\begin{cases} x = 2t-1 \\ y = t^3 \end{cases}$

6 Logarithmic Differentiation

Taking the derivatives of some complicated functions can be simplified by using logarithms. This is called **logarithmic differentiation**.

Example Differentiate the function $y = \frac{x^5}{(1-10x)\sqrt{x^2+2}}$

Solution

Taking logarithms of both sides we obtain

$$\ln y = \ln\frac{x^5}{(1-10x)\sqrt{x^2+2}}$$

$$\ln y = \ln x^5 - \ln(1-10x) - \ln\sqrt{x^2+2}$$

Differentiate both sides with respect to x to get

$$\frac{y'}{y} = \frac{5x^4}{x^5} + \frac{10}{1-10x} - \frac{\left(\sqrt{x^2+2}\right)'}{\sqrt{x^2+2}}$$

$$\frac{y'}{y} = \frac{5x^4}{x^5} + \frac{10}{1-10x} - \frac{x}{x^2+2}$$

Soving for y'

$$y' = y\left(\frac{5x^4}{x^5} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

$$= \frac{x^5}{(1-10x)\sqrt{x^2+2}}\left(\frac{5x^4}{x^5} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)$$

We can also use logarithmic differentiation to differentiate functions in the form

$$y = \left[u(x)\right]^{v(x)}$$

Example Differentiate $y = x^x$, $y = x^{x^x}$, $y = a^x$ where a is a constant.

7 Higher Order Derivatives

7-1 Definition of Higher Order derivatives

A derivative of the second order, or the second derivative, of a function $y = f(x)$ is the derivative of its derivative; that is

$$y'' = (y')'$$

The second derivative may be denoted as

$$y'',\ \text{or}\ \frac{d^2y}{d^2x},\ \text{or}\ f''(x)$$

Generally, the nth derivative of a function $y = f(x)$ is the derivative of the derivative of order (n-1) . For the nth derivative we use the notation

$$y^{(n)},\ \text{or}\ \frac{d^ny}{dx^n},\ \text{or}\ f^{(n)}(x)$$

Example Find the second derivative of the function $y = \ln(1-x)$

Solution

$$y' = \frac{-1}{1-x},\ y'' = \left(\frac{-1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

7-2 Higher-Order Derivatives of functions represented Parametrically

If

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

then the derivative $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2},...$ can successively be calculated by the formulas

$$y'_x = \frac{y'_t}{x'_t}, y''_{xx} = (y'_x)'_x = \frac{(y'_x)'_t}{x'_t}\ \text{and so forth.}$$

For the second derivative we have the formula

$$y''_{xx} = \frac{x'_t y''_{tt} - x''_{tt} y'_t}{(x'_t)^3}$$

Example Find y'' if $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$. Answer: $-\frac{b}{a^2 \sin^3 t}$.

8 Differential

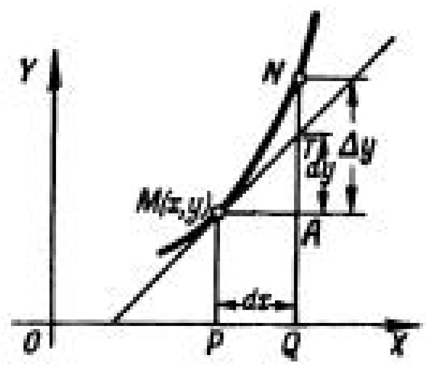
8-1 First-Order Differential

The differential of a function $y = f(x)$ is the principal part of it increment, which is linear relative to the increment $\Delta x = dx$ of the independent variable x. The differential of a function is equal to the product of it derivative by the differential of the independent variable

$$dy = y'dx$$

whence

$$y' = \frac{dy}{dx}$$



8-2 Properties of Differential

1. $dc = 0$, c is a constant
2. $d(cu) = cdu$
3. $d(u \pm v) = du \pm dv$
4. $d(uv) = udv + vdu$
5. $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$ $v \neq 0$
6. $df(u) = f'(u)du$

8-3 Approximation by Differential

For the function $y = f(x)$, $\Delta y \approx dy$; that is $f(x+\Delta x) - f(x) \approx f'(x)\Delta x$ whence $f(x+\Delta x) \approx f'(x)\Delta x + f(x)$

8-4 Higher-Order Differential

If $y = f(x)$ and x is the independent variable, then

$$d^2 y = y''(dx)^2$$

$$d^3 y = y'''(dx)^3$$

....

$$d^n y = y^{(n)}(dx)^n$$

9 Theorems Relative to Derivative

9-1 Rolle's Theorem

If $f(x)$ is continuous on the interval $[a,b]$, differentiable at every interior point of the interval and $f(a) = f(b)$, then there exist at least a point $x = \xi$, $a < \xi < b$ where $f'(\xi) = 0$.

Proof

If *f* is continuous on the interval $[a,b]$, then it attains on the interval a relative maximum value *M* and a minimum value *m*. If *m*=*M*, then *f* is constant, say, $f(x) = m$, implying that $f'(\xi) = 0$. If *m*≠*M*, we suppose that *M* >0 and *f* attains the maximum value *M* at $x = \xi$, that is $f(\xi) = M$, $\xi \neq a, b$. If $f(\xi)$ is the upper bound of *f*, then $f(\xi + h) - f(\xi) \leq 0$ and therefore,

$$\frac{f(\xi+h) - f(\xi)}{h} \leq 0, \quad h > 0 \Rightarrow \lim_{h \to 0} \frac{f(\xi+h) - f(\xi)}{h} \leq 0$$

$$\frac{f(\xi+h) - f(\xi)}{h} \geq 0, \quad h < 0 \Rightarrow \lim_{h \to 0} \frac{f(\xi+h) - f(\xi)}{h} \geq 0$$

Hence $f'(\xi) = 0$.

Example Consider the function $f(x) = \sin x$ The function is both continuous and differentiable everywhere, hence it is continuous on $[0, 2\pi]$ and differentiable on (a, b). Moreover

$$f(0) = \sin 0 = 0, \; f(2\pi) = \sin 2\pi = 0$$

so that f satisfies the hypotheses of Rolle's theorem on the interval $[0, 2\pi]$. Since $f'(c) = \cos c$. Rolle's theorem guarantees that there is at least one point in $(0, 2\pi)$ such that

$$\cos c = 0$$

which yields two values for *c*, namely $c_1 = \pi/2$ and $c_2 = 3\pi/2$

Example Verify that the hypotheses of Rolle's theorem is satisfied on the given interval and find all values of *c* that satisfy the conclusion of the theorem

$$f(x) = \frac{x^2 - 1}{x - 2}, [-1,1]$$

Solution

On the interval $[-1,1]$ $f(x)$ is continuous and it is differentiable on $(-1,1)$

$$f'(x) = \frac{(x^2-1)'(x-2)-(x-2)'(x^2-1)}{(x-2)^2} = \frac{2x(x-2)-(x^2-1)}{(x-2)^2} = \frac{2x^2-4x-x^2+1}{(x-2)^2}$$

$$f'(\xi) = 0$$

$$\xi^2 - 4\xi + 1 = 0$$

which has the roots $\xi_1 = 2-\sqrt{3}, \xi_2 = 2+\sqrt{3}$

Hence $\xi = 2-\sqrt{3}$ satisfies the theorem.

9-2 Mean-Value Theorem

Let f be differentiable on (a,b) and continuous on $[a,b]$. Then there is at least one point ξ in (a,b) such that $f(b)-f(a) = f'(\xi)(b-a)$.

Proof

The slope of $g(x)$ is $Q = \frac{f(b)-f(a)}{b-a}$

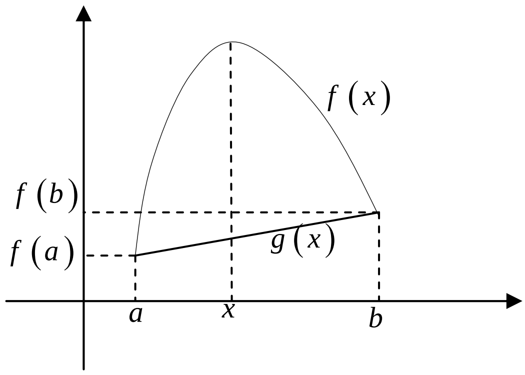
since $g(x)$ passes through the point $(a, f(a))$,

then the equation of the line is defined by

$$g(x) - f(a) = Q(x-a)$$

then $g(x) = f(a) + Q(x-a)$.

Let



$$F(x) = f(x) - g(x) = f(x) - [f(a) + Q(x-a)] = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Hence we obtain the function $F(x)$ which is continuous on $[a,b]$, differentiable on (a,b) and $F(a) = F(b) = 0$. By Rolle's Theorem, $\ni \xi \in (a,b)$ such that $F'(\xi) = 0$.

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$F'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a}$$

then $f'(\xi) - \frac{f(b)-f(a)}{b-a} = 0$. Hence $f(b)-f(a) = f'(\xi)(b-a)$ ■

Example Let $f(x) = x^3 + 1$. Show that f is satisfies the hypotheses of the Mean-Value Theorem on the interval $[1,2]$ and find all values of ξ in this interval whose existence is guaranteed by the theorem.

Solution

Because $f(x)$ is a polynomial, f is continuous and differentiable everywhere, hence is continuous on $[1,2]$ and differentiable on $(1,2)$. Thus, the hypotheses of the Mean-Value Theorem are satisfied with $a=1$ and $b=2$. But

$f(a)=f(1)=2$, $f(b)=f(2)=9$

$f'(x)=3x^2$, $f'(c)=3c^2$

so that the equation

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}$$

$\Leftrightarrow 3\xi^2=7$ which has two solutions

$\xi=\sqrt{7/3}$ and $\xi=-\sqrt{7/3}$

So $\xi=\sqrt{7/3}$ is the number whose existence is guaranteed by the Mean-Value Theorem.

9-3 Cauchy's Theorem

Let $f(x)$ and $g(x)$ be continuous and differentiable function over the interval $[a,b]$ and $g'(x)\neq 0$ over $[a,b]$. Then there exists an interior point $x=\xi$ to the interval $[a,b]$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$

Proof

Let define Q by $Q=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

Notice that $g(b)-g(a)\neq 0$ since if not, $g(b)=g(a)$ then by Rolle's Theorem, $g'(x)=0$ at a point interior to $[a,b]$. It contradicts to the condition of the theorem.

Let form a function $F(x)=f(x)-f(a)-Q\left[g(x)-g(a)\right]$, which satisfies the condition of the Rolle's Theorem, then there exists a number $x=\xi, a<\xi<b$, such that $F'(\xi)=0$. Since $F'(x)=f'(x)-Qg'(x)$, then $F'(\xi)=f'(\xi)-Qg'(\xi)=0 \Rightarrow Q=\dfrac{f'(\xi)}{g'(\xi)}$. Hence

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)} \blacksquare$$

9-4 L' Hopital's Rule

Consider the function $F(x)=f(x)/g(x)$, where both $f(x)=0$ and $g(x)=0$ when $x=a$. Then, for any $x>a$ there exists a value ξ, $a<\xi<x$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$$

or
$$\frac{f(x)}{g(x)}=\frac{f'(\xi)}{g'(\xi)}$$

Now as $x\to a, \xi\to a$, therefore when the limit exists

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{\xi\to a}\frac{f'(x)}{g'(x)}$$

This result is known as l' Hopital's Rule and is usually written as

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example Evaluate $\lim_{x\to 0}\frac{\cos x-1}{x^2-x}$

L'Hopital's Rule can still be applied in cases where $f(x)\to\infty$ and $g(x)\to\infty$ when $x\to a$, simply by writing

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{1/f(x)}{1/g(x)}$$

Now $1/f(x)\to 0$ and $1/g(x)\to 0$ as $x\to a$ and the rule applies. Therefore,

$$L=\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{1/g(x)}{1/f(x)}=\lim_{x\to a}\left[\frac{-g'(x)}{[g(x)]^2}\Big/\frac{-f'(x)}{[f(x)]^2}\right]$$

$$=\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]^2\left[\frac{g'(x)}{f'(x)}\right]=L^2\lim_{x\to a}\frac{g'(x)}{f'(x)}$$

Hence

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Similarly, if $f(x)$ and $g(x)$ both tend to zero, or both tend to infinity as *x* tend to infinity

the rule applies. By writing $x=1/u$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{u\to 0}\frac{f(1/u)}{g(1/u)}=\lim_{u\to 0}\left\{-\frac{1}{u^2}f'\left(\frac{1}{u}\right)\Big/-\frac{1}{u^2}g'(1/u)\right\}$$

$$=\lim_{u\to 0}\left\{f'(1/u)/g'(1/u)\right\}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

If, after one application of l' Hopital's rule the limit is still indeterminate, the process can be repeated until a determinate form is reached.

Example Evaluate

(i) $\lim_{x\to 0}\frac{\sin^2 x}{x^3+2x^2}$ (ans: $1/2$) **(ii)** $\lim_{x\to\infty}x^3e^{-x^2}$ (ans: 0)

9-5 Taylor's Theorem for Functions of One Variable

Suppose that the function $y=f(x)$ has ($n+1$)th order derivative in the neighborhood of the point $x=a$. We will find the polynomial of order n at most such that

$$P_n(a)=f(a),P_n'(a)=f'(a),P_n''(a)=f''(a),\ldots,P_n^{(n)}(a)=f^{(n)}(a)$$

The sought-for polynomial is of the form

$$P_n(x)=C_0+C_1(x-a)+C_2(x-a)^2+C_3(x-a)^3+\cdots+C_n(x-a)^n$$

Let calculate the nth derivative of $P_n(x)$

$$P_n'(x)=C_1+2C_2(x-a)+3C_3(x-a)^2+\cdots+nC_n(x-a)^{n-1}$$

$$P_n''(x)=2C_2+6C_3(x-a)+\cdots+n(n-1)C_n(x-a)^{n-2}$$

$$P_n'''(x)=6C_3+4\cdot 3\cdot 2C_4(x-a)+\cdots+n(n-1)(n-2)C_n(x-a)^{n-3}$$

$$\vdots$$

$$P_n^{(n)}(x) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot C_n$$

Then we can obtain

$$f(a) = C_0$$
$$f'(a) = C_1$$
$$f''(a) = 2\cdot 1\cdot C_2$$
$$\vdots$$
$$f^{(n)}(a) = n(n-1)(n-2)\cdots 2\cdot 1\cdot C_n$$

and hence

$$C_0 = f(a)$$
$$C_1 = f'(a)$$
$$C_2 = \frac{1}{1\cdot 2} f''(a)$$
$$C_3 = \frac{1}{1\cdot 2\cdot 3} f'''(a)$$
$$\vdots$$
$$C_n = \frac{1}{n!} f^{(n)}(a)$$

Therefore, we obtain

$$P_n(x) = f(a) + \frac{x-a}{1} f'(a) + \frac{(x-a)^2}{1\cdot 2} f''(a) + \frac{(x-a)^3}{1\cdot 2\cdot 3} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

Let $R_n(x)$ be the difference between the function $f(x)$ and the polynomial $P_n(x)$; that is,

$$R_n(x) = f(x) - P_n(x)$$

Then

$$f(x) = P_n(x) + R_n(x)$$

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x)$$

$R_n(x)$ is called the *remainder* and is defined by

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} Q(x)$$

where $Q(x)$ is the function to be defined.

Now we have

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} Q(x)$$

we will find $Q(x)$.

Consider an auxiliary function $F(t), a < t < x$ which is define as

$$F(t) = f(x) - f(t) - \frac{x-t}{1} f'(t) - \frac{(x-t)^2}{2!} f''(t) - \cdots - \frac{(x-t)^n}{n!} f^{(n)}(t) - \frac{(x-t)^{n+1}}{(n+1)!} Q$$

By computing $F'(t)$ and simplifying, we obtain

$$F'(t) = -f'(t) + f'(t) - \frac{x-t}{1} f''(t) + \frac{2(x-t)}{2!} f''(t) - \frac{(x-t)^2}{2!} f'''(t)$$

$$\cdots - \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) + \frac{n(n-t)^{n-1}}{n!} f^{(n)}(t) - \frac{(x-t)^n}{n!} f^{(n+1)}(t) + \frac{(n+1)(x-t)^n}{(n+1)!} Q$$

$$F'(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t) + \frac{(x-t)^n}{n!} Q$$

We can see that the function $F(t)$ satisfies the condition of Rolle's Theorem, then there exists a number ξ, $a < \xi < x$ such that $F'(\xi) = 0$. Then

$$-\frac{(x-\xi)^n}{n!} f^{(n+1)}(\xi) + \frac{(x-\xi)^n}{n!} Q = 0$$

$$Q = f^{(n+1)}(\xi)$$

and thus,

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

which is called *Lagrange formula for the remainder.* Since ξ is between x and a we can write it in the form

$$\xi = a + \theta(x-a)$$

where θ is between 0 and 1; that is $0 < \theta < 1$. Then the remainder can be written as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}\left[a + \theta(x-a)\right]$$

The formula

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$+ \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}\left[a + \theta(x-a)\right], 0 < \theta < 1$$

is called *Taylor Formula* for the function $f(x)$. If , in this formula, $a = 0$ we obtain

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x)$$

which is known as *Maclaurin Formula.*

Example: Use Macluarin Formula to expand the functions $e^x, \sin x$, and $\cos x$.

Exercises

Exercise 1 through 4, use definition of derivative

1 Given $y = f(x) = x^2 + 5x - 8$, find Δy and $\Delta y/\Delta x$ as x changes (a) from $x_0 = 1$ to $x_1 = x_0 + \Delta x = 1.2$ (b) and $x_0 = 1$ to $x_1 = 0.8$.

2 Find $\Delta y/\Delta x$, given $y = x^3 - x^2 - 4$. Find also the value of $\Delta y/\Delta x$ when (a) $x = 4$, (b) $x = 0$, (c) $x = -1$.

3 Find the derivative of $y = f(x) = \dfrac{1}{x-2}$ at $x = 1$ and $x = 3$.

4 Find the derivative of $f(x) = \dfrac{2x-3}{3x+4}$

5 Differentiate

(a) $y=\dfrac{3-2x}{3+2x}$, (b) $y=\dfrac{x^2}{\sqrt{4-x^2}}$

6 Find $\dfrac{dy}{dx}$, given $x=y\sqrt{1-y^2}$

Find the derivative of the following functions

7 $f(x)=x\cot x$

8 $y=\tan x-\cot x$

9 $f(x)=x\sin^{-1}x$

10 $y=\dfrac{\sin x+\cos x}{\sin x-\cos x}$

11 $y=\dfrac{(1+x^2)\tan^{-1}x-x}{2}$

12 $y=2t\sin t-(t^2-2)\cos t$

13 $f(x)=\arctan x+\text{arc}\cot x$

14 $y=x^7e^x$

15 $y=\dfrac{x^2}{\ln x}$

16 $y=e^x\arcsin x$

17 $y=x\sinh x$

18 $y=\tan^{-1}x-\tanh^{-1}x$

19 $y=\dfrac{x^2}{\cosh x}$

20 $y=\sin^{-1}x\sinh^{-1}x$

21 $y=\tanh x-x$

22 $y=\dfrac{\cosh^{-1}x}{x}$

Derivative of composite function

23 $f(x)=(1+3x-5x^2)^{30}$

24 $y=\sqrt{1-x^2}$

25 $f(x)=(3-2\sin x)^4$

26 $y=\tan x-\dfrac{1}{3}\tan^3 x+\dfrac{1}{5}\tan^5 x$

27 $y=\sqrt{\cot x}$

28 $y=\sqrt[3]{\sin^2 x}+\dfrac{1}{\cos^3 x}$

29 $y=\csc^2 x+\sec^2 x$

30 $y=\sqrt{1+\sin^{-1}x}$

31 $y=\dfrac{1}{\tan^{-1}x}$

32 $y=\dfrac{1}{3\cos^3 x}-\dfrac{1}{\cos x}$

33 $y=\sin(x^2-5x+1)+\tan\dfrac{\alpha}{x}$

34 $y=\dfrac{1+\cos 2x}{1-\cos 2x}$

35 $f(t)=\sin t\sin(t+\phi)$

36 $y=\sin^{-1}2x$

37 $y=\sin^{-1}\dfrac{1}{x^2}$

38 $y=\cos^{-1}e^x$

39 $y=\ln(2x+7)$

40 $y=\ln^2 x-\ln(\ln x)$

41 $y=\tan^{-1}(\ln x)+\ln(\tan^{-1}x)$

42 $y=\sqrt{\ln x+1}+\ln(\sqrt{x}+1)$

43 $y=(a+x)\sqrt{a-x}$

44 $y=\ln\left(\sqrt{1+e^x}-1\right)-\ln\left(\sqrt{1+e^x}+1\right)$

45 $y=\sin^2(t^3)$

46 $y=\arcsin x^2+\arccos x^2$

47 $y=\sin^{-1}\dfrac{x^2-1}{x^2}$

48 $y=\dfrac{\arccos x}{\sqrt{1-x^2}}$

49 $y=\arcsin\dfrac{x}{\sqrt{1+x^2}}$

50 $y=\sqrt{a^2-x^2}+a\arcsin\dfrac{x}{a}$

51 $y=x\sqrt{a^2-x^2}+a^2\arcsin\dfrac{x}{a}$

52 $y=\ln\left(x+\sqrt{a^2+x^2}\right)$

53 $y=\dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$

54 $f(x)=\dfrac{x\arcsin x}{\sqrt{1-x^2}}+\ln\sqrt{1-x^2}$

55 $y=\cosh^{-1}\ln x$

56 $y=\tanh^3 2x$

57 Given the function $f\left(x\right)=e^{-x}$, determine $f\left(0\right)+xf'\left(0\right)$.

58 Given the function $f\left(x\right)=\sqrt{1+x}$, calculate the expression $f\left(3\right)+\left(x-3\right)f'\left(3\right)$

59 Given $f\left(x\right)=\tan x, g\left(x\right)=\ln\left(1-x\right)$, calculate $\dfrac{f'(0)}{g'(0)}$

60 Show that the function $y=xe^{-x}$ satisfies the equation $xy'=\left(1-x\right)y$

61 Show that the function $y=xe^{-\frac{x^2}{2}}$, satisfies the equation $xy'=\left(1-x^2\right)y$

62 Show that the function $y=\dfrac{1}{1+x+\ln x}$, satisfies the equation $xy'=y\left(y\ln x-1\right)$

Logarithmic Differentiation

63 $y=\left(x+1\right)\left(2x+1\right)\left(3x+1\right)$

64 $y=\sqrt[x]{x}$

65 $y=\dfrac{\left(x+1\right)^2}{\left(x+1\right)^2\left(x+3\right)^4}$

66 $y=x^{\sqrt{x}}$

67 $y=\sqrt{\dfrac{x\left(x-1\right)}{x-2}}$

68 $y=x^x$

69 $y=x^{x^2}$

70 $y=x^{\sin x}$

71 $y=\left(1+\dfrac{1}{x}\right)^x$

72 $y=\left(\arctan x\right)^x$

y is the function of *x* and determined in parametric form. Find $y'=\dfrac{dy}{dx}$

73 $\begin{cases} x=2t-1 \\ x=t^3 \end{cases}$

74 $\begin{cases} x=a\cos^2 t \\ y=a\sin^2 t \end{cases}$

75 $\begin{cases} x=\dfrac{1}{t+1} \\ y=\left(\dfrac{t}{t+1}\right)^2 \end{cases}$

76 $\begin{cases} x=\dfrac{2at}{1+t^2} \\ y=\dfrac{a\left(1-t^2\right)}{1+t^2} \end{cases}$

78 $\begin{cases} x=\sqrt{t} \\ y=\sqrt[3]{t} \end{cases}$

80 $\begin{cases} x=\arccos\dfrac{1}{\sqrt{1+t^2}} \\ y=\arcsin\dfrac{t}{\sqrt{1+t^2}} \end{cases}$

81 $\begin{cases} x=e^{-t} \\ y=e^{2t} \end{cases}$

82 Compute $\dfrac{dy}{dx}$ for $t=\dfrac{\pi}{2}$, if $\begin{cases} x=a\left(t-\sin t\right) \\ y=a\left(1-\cos t\right) \end{cases}$

83 Show that the function *y* given in the parametric form by the equations

$$\begin{cases} x=2t+3t^2 \\ y=t^2+2t^3 \end{cases}$$

satisfies the equation

$$y=\left(\frac{dy}{dx}\right)^2+2\left(\frac{dy}{dx}\right)^3$$

Find the derivative $y'=\dfrac{dy}{dx}$ of the implicit function y

84 $a\cos^2(x+y)=b$

85 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

86 $\tan y=xy$

87 $xy=\arctan\dfrac{x}{y}$

88 $\sqrt{x}+\sqrt{y}=\sqrt{a}$

89 $e^y=x+y$

90 $\ln x+e^{-\frac{y}{x}}=c$

91 $\sqrt{x^2+y^2}=c\arctan\dfrac{y}{x}$

92 $y^x=x^y$

Find the derivatives y' of specified functions y *at the indicated points*

93 $(x+y)^3=27(x-y)$ for $x=2$ and $y=1$

94 ye^y+e^{x+1} for $x=0$ and $y=1$

95 $y^2=x+\ln\dfrac{y}{x}$ for $x=1$ and $y=1$

96 Find $y^{(6)}$ of the function $y=\sin 2x$

97 Show that the function $y=e^{-x}\cos x$ satisfied the differential equation $y^{(4)}+4y=0$

98 Find the nth derivatives of the functions

a) $y=\dfrac{1}{1-x}$ **b)** $y=\sqrt{x}$ **c)** $y=\dfrac{1}{1+x}$ **d)** $y=\ln(1+x)$ **e)** $y=\dfrac{1+x}{1-x}$

f) $y=\ln(1+x)$ **g)** $y=xe^x$

99 In the following problem find $\dfrac{d^2y}{dx^2}$

a) $\begin{cases}x=\ln t\\ y=t^3\end{cases}$ **b)** $\begin{cases}x=\arctan t\\ y=\ln(1+t^2)\end{cases}$ **c)** $\begin{cases}x=\arcsin t\\ y=\sqrt{1-t^2}\end{cases}$ **d)** $\begin{cases}x=a\cos t\\ y=a\sin t\end{cases}$

e) $\begin{cases}x=e^{-at}\\ y=e^{at}\end{cases}$ **f)** $\begin{cases}x=\ln t\\ y=\dfrac{1}{1-t}\end{cases}$

100 Use L'Hopital Rule to find the limits

a) $\lim\limits_{x\to1}\dfrac{1-x}{1-\sin\frac{\pi x}{2}}$ **b)** $\lim\limits_{x\to0}\dfrac{\cosh x-1}{1-\cos x}$ **c)** $\lim\limits_{x\to0}\dfrac{\tan x-\sin x}{x-\sin x}$ **d)** $\lim\limits_{x\to0}\dfrac{\frac{\pi}{x}}{\cot\frac{\pi x}{2}}$

e) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\tan x}{\tan 5x}$ **f)** $\lim\limits_{x\to0}\dfrac{\ln(\sin mx)}{\ln(\sin x)}$ **g)** $\lim\limits_{x\to1}(1-x)\tan\dfrac{\pi x}{2}$ **h)** $\lim\limits_{x\to1}\ln x\ln(x-1)$

i) $\lim\limits_{x\to+\infty}x^{\frac{1}{x}}$ **j)** $\lim\limits_{x\to1}x^{\frac{1}{1-x}}$ **k)** $\lim\limits_{x\to0}x^{\sin x}$ **l)** $\lim\limits_{x\to1}(1-x)^{\cos\frac{\pi x}{2}}$

101 Find the approximate values of the followings using the formula

$f(x+\Delta x)\approx f'(x)\Delta x+f(x)$

a) $\cos 61^\circ$ **b)** $\ln 0.9$ **c)** tan44º **d)** arctan 1.05 **e)** $e^{0.2}$

102 Approximate the functions

a) $f(x)=\sqrt{1+x}$ for $x=0.2$ **b)** $y=e^{1-x^2}$ for $x=1.05$ **c)** $f(x)=\sqrt[3]{\dfrac{1-x}{1+x}}$ for $x=0.1$

103 $u=\sqrt{1-x^2}$, find d^2u . find d^2y .

104 $y=\arccos x$, find d^2y .

105 $y=\sin x\ln x$, find d^2y .

106 $f(x)=x-x^3$ on the intervals $-1\le x\le 0$ and $0\le x\le 1$ satisfies the Rolle theorem. Find the appropriate values of ξ .

107 Test whether the Mean-Value theorem holds for the function $f(x)=x-x^3$ on the interval $[-2,1]$ and find the appropriate value of ξ .

108 **a)** For the function $f(x)=x^2+2$ and $g(x)=x^3-1$. Test whether the Cauchy theorem holds on the interval $[1,2]$ and find ξ .

b) do the same with respect to $f(x)=\sin x$ and $g(x)=\cos x$.

109 Verify the following by Taylor's formula

a) $e^x=e^a\left[1+(x-a)+\dfrac{(x-a)^2}{2!}+\dfrac{(x-a)^3}{3!}+\cdots\right]$

b) $\sin x=\sin a+(x-a)\cos a-\dfrac{(x-a)^2}{2!}\sin a-\dfrac{(x-a)^3}{3!}\cos a+\cdots$

c) $\cos x=\cos a-(x-a)\sin a-\dfrac{(x-a)^2}{2!}$

d) $\ln(a+x)=\ln a+\dfrac{x}{a}-\dfrac{x^2}{2a^2}+\dfrac{x^3}{3a^3}+\cdots$

110 Expand ln$x$ in powers of $(x-2)$ to four terms.

111 Expand $\tan x$ in powers of $\left(x-\dfrac{\pi}{4}\right)$ to three terms

112 Expand $\sin x$ in powers of $\left(\dfrac{\pi}{6}+x\right)$ to four terms.

Indefinite Integral

1 Antiderivative or Indefinite Integral

Problem: Given a function $f(x)$, find a function $F(x)$ whose derivative is equal to $f(x)$; that is $F'(x) = f(x)$.

Definition1

We call the function $F(x)$ a antiderivative of the function $f(x)$ on the interval $[a,b]$ if $F'(x) = f(x), \forall x \in [a,b]$.

Definition2

We call *indefinite integral* of the function *f*, which is denoted by $\int f(x)dx$, all the expressions of the form $F(x)+C$ where $F(x)$ is a primitive of $f(x)$. Hence, by the definition we have

$$\int f(x)dx = F(x)+C$$

C is called the constant of integration. It is an abitrary constant.
From the definition 2 we obtain

1. If $F'(x) = f(x)$, then $\left(\int f(x)dx\right)' = \left(F(x)+C\right)' = f(x)$
2. $d\left(\int f(x)dx\right) = f(x)dx$
3. $\int dF(x) = F(x)+C$

2 Table of Integrals

1. $\int x^r dx = \frac{x^{r+1}}{r+1} + C, r \neq -1$
2. $\int \frac{dx}{x} = \ln|x| + C$
3. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C = -\frac{1}{a}arc\cot\frac{x}{a} + C, (a \neq 0)$
4. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C, (a \neq 0)$
5. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, (a \neq 0)$
6. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$
7. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C, (a > 0)$
8. $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + c$
9. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c$
10. $\int a^x dx = \frac{a^x}{\ln a} + C, (a > 0)$
11. $\int e^x dx = e^x + C$

12. $\int \sin x dx = -\cos x + C$

13. $\int \cos x dx = \sin x + C$

14. $\int \sinh x dx = \cosh x + c$

15. $\int \cosh x = \sinh x + c$

16. $\int \frac{dx}{\cosh^2 x} = \tanh x + c$

17. $\int \frac{dx}{\cos^2 x} = \tan x + C$

18. $\int \frac{dx}{\sinh^2 x} = -\coth x + c$

19. $\int \frac{dx}{\sin^2 x} = -\cot x + C$

20. $\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C$

21. $\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C = \ln|\tan x + \sec x| + C$

3. Some Properties of Indefinite Integrals

Linearity

1. $\int\left[f_1(x) + f_2(x) + \cdots + f_n(x)\right] = \int f_1(x)dx + \int f_2(x)dx + \cdots + \int f_n(x)dx$

2. If a is a constant, then $\int af(x)dx = a\int f(x)dx$

Moreover,

3. If $\int f(x)dx = F(x) + C$, then $\int f(ax)dx = \frac{1}{a}F(ax) + C$

4. If $\int f(x)dx = F(x) + C$, then $\int f(x+b)dx = F(x+b) + C$

5. If $\int f(x)dx = F(x) + C$, then $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

Example 1

1. $\int\left(2x^3 - 3\sin x + 5\sqrt{x}\right)dx$ ans: $\frac{1}{2}x^4 + 3\cos x + \frac{10}{3}x\sqrt{x} + C$

2. $\int\left(\frac{3}{\sqrt[3]{x}} + \frac{1}{2\sqrt{x}} + x\sqrt[4]{x}\right)dx$ ans: $\frac{9}{2}\sqrt[3]{x^2} + \sqrt{x} + \frac{4}{9}x^2\sqrt[4]{x} + C$

3. $\int \frac{dx}{x+3}$ ans: $\ln|x+3| + C$

4. $\int \cos 7x dx$ ans: $\frac{1}{7}\sin(7x) + c$

5. $\int \sin(2x-5)dx$ ans: $-\frac{1}{2}\cos(2x-5) + c$

4 Integration By Substitution

4.1 Change of Variable in an Indefinite Integral

Putting $x = \varphi(t)$ where t is a new variable and φ is a continuously differentiable function, we obtain

$$\int f(x)\,dx = \int f[\varphi(t)]\varphi'(t)\,dt \qquad (1)$$

The attempt is made to choose the function φ in such a way that the right side of (1) becomes more convenient for integration.

Example 1 Evaluate the integral $I = \int x\sqrt{x-1}dx$

Solution

Putting $t = \sqrt{x-1}$, whence $x = t^2 + 1$ and $dx=2tdt$. Hence,

$$\int x\sqrt{x-1}dx = \int (t^2+1)t\cdot 2tdt = 2\int (t^4+t^2)dt = \tfrac{2}{5}t^5 + \tfrac{2}{3}t^3$$

$$= \tfrac{2}{5}(x-1)^{\frac{5}{2}} + \tfrac{2}{3}(x-1)^{\frac{3}{2}} + c$$

Sometimes substitution of the form $u = \varphi(x)$ are used. Suppose we succeeded in transorming the integrand $f(x)dx$ to the form

$$f(x)dx = g(u)du$$

where $u = \varphi(x)$. If $\int g(u)du$ is known, that is,

$$\int g(u)du = F(u) + k,$$

then

$$\int f(x)dx = F[\varphi(x)] + c$$

Example 2 Evaluate (1) $\int \frac{dx}{\sqrt{5x-2}}$ (2) $\int x^2 e^{x^3} dx$

Solution

Putting $u = 5x-2$; $du = 5dx; dx = \frac{1}{5}du$, we obtain (1)

$$\int \frac{dx}{\sqrt{5x-2}} = \frac{1}{5}\int \frac{du}{\sqrt{u}} = \frac{1}{5}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{2}{5}\sqrt{5x-2} + c$$

4.2 Trigonometric Substitutions

1) If the integral contains the radical $\sqrt{a^2-x^2}$, we put $x = a\sin t$; whence

$$\sqrt{a^2-x^2} = a\cos t$$

2) If the integral contains the radical $\sqrt{x^2-a^2}$, we put $x = a\sec t$ whence

$$\sqrt{x^2-a^2} = a\tan t$$

3) If the integral contains the radical $\sqrt{x^2+a^2}$, we put $x = a\tan t$ whence

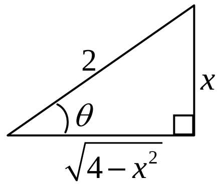
$$\sqrt{x^2+a^2} = a\sec t$$

We summarize in the the trigonometric substitution in the table below.

Expression in the integrand	Substitution	Identities needed
$\sqrt{a^2-x^2}$	$x = a\sin t$	$a^2 - a^2\sin^2 t = a^2\cos^2 t$
$\sqrt{a^2+x^2}$	$x = a\tan t$	$a^2 + a^2\tan^2 t = a^2\sec^2 t$
$\sqrt{x^2-a^2}$	$x = a\sec t$	$a^2\sec^2 t - a^2 = a^2\tan^2 t$

Example 3 Evaluate $I=\int \frac{dx}{x^2\sqrt{4-x^2}}$

Solution



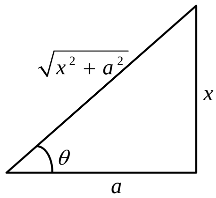
Let $x=2\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ $\Rightarrow dx=2\cos\theta d\theta$

$$I=\int\frac{2\cos\theta d\theta}{4\sin^2\theta\sqrt{4\cos^2\theta}}=\int\frac{2\cos\theta d\theta}{4\sin^2\theta\cdot 2\cos\theta}=\frac{1}{4}\int\frac{d\theta}{\sin^2\theta}$$

$$=\frac{1}{4}\int\csc^2\theta d\theta=-\frac{1}{4}\cot\theta+C=-\frac{1}{4}\cdot\frac{\sqrt{4-x^2}}{x}+C$$

Example 4 $I=\int\frac{dx}{\sqrt{x^2+a^2}}$

Solution



$x=a\tan\theta,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$ $\qquad dx=a\sec^2\theta d\theta$

$$I=\int\frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta+a^2}}$$

$$=\int\frac{a\sec^2\theta d\theta}{a|\sec\theta|}=\int\sec\theta d\theta=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C$$

$$=\ln\left|\sqrt{x^2+a^2}+x\right|-\ln a+C=\ln\left|\sqrt{x^2+a^2}+x\right|+C_1$$

Example 5 Evaluate $\int\frac{\sqrt{x^2-25}}{x}dx$

Solution

Let $x=5\sec\theta$

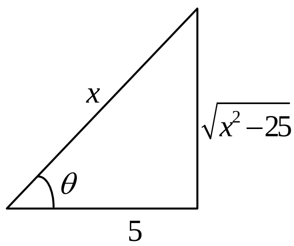
$\frac{dx}{d\theta}=\sec\theta\tan\theta$ or $dx=5\sec\theta\tan\theta d\theta$

Thus,

$$\int\frac{\sqrt{x^2-25}}{x}dx=\int\frac{\sqrt{25\sec^2\theta-25}}{5\sec\theta}(5\sec\theta\tan\theta)d\theta$$

$$=\int\frac{5|\tan\theta|}{5\sec\theta}(5\sec\theta\tan\theta)d\theta=5\int\tan^2\theta d\theta$$

$$=5\int(\sec^2\theta-1)d\theta=5\tan\theta-5\theta+C$$



We obtain $\tan\theta=\frac{\sqrt{x^2-25}}{5}$. Hence $\int\frac{\sqrt{x^2-25}}{x}dx=\sqrt{x^2-25}-5\sec^{-1}\left(\frac{x}{5}\right)+C$

Example 6 Evaluate $\int\frac{\sqrt{x^2+1}}{x^2}dx$

5. Integration by Parts

Suppose that u and v are differentiable function of x, then

$$d(uv) = udv + vdu$$

By integrating, we obtain

$$uv = \int udv + \int vdu$$

Or

$$\int udv = uv - \int vdu$$

Example

1. $\int x\sin x dx$ (let $u = x$) ans: $-x\cos x + \sin x + C$
2. $\int \arctan x dx$ (let $u = \arctan x$) ans: $x\arctan x - \frac{1}{2}\ln\left|1+x^2\right| + C$
3. $\int x^2 e^x dx$ (let $u = x^2$) ans: $e^x\left(x^2 - 2x + 2\right) + C$
4. $\int \left(x^2 + 7x - 5\right)\cos 2x dx$ ans: $\left(x^2+7x-5\right)\frac{\sin 2x}{2} + \left(2x+7\right)\frac{\cos 2x}{4} - \frac{\sin 2x}{4} + C$

6 Standard Integrals Containing a Quadratic Trinomial

6.1 Integrals of the form $\int \frac{mx+n}{ax^2+bx+c}dx$ or $\int \frac{mx+n}{\sqrt{ax^2+bx+c}}dx$ where $b^2 - 4ac < 0$

We proceed the calculation by completing square the trinomial and then use the appropriate formulas or substitutions.

Example 1

1. $\int \frac{dx}{x^2 - 2x + 5}$ ans: $\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$
2. $\int \frac{dx}{2x^2 + 8x + 20}$ ans: $\frac{1}{2\sqrt{6}}\arctan\frac{x+2}{\sqrt{6}} + C$
3. $\int \frac{x}{x^2 - 4x + 8}dx$ ans: $\frac{1}{2}\ln\left[(x-2)^2 + 4\right] + \tan^{-1}\left(\frac{x-2}{2}\right) + c$
4. $\int \frac{x+3}{x^2 - 2x + 5}dx$ ans: $\frac{1}{2}\ln\left(x^2 - 2x + 5\right) + 2\arctan\frac{x-1}{2} + C$
5. $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$ ans: $5\sqrt{x^2+4x+10} - 7\ln\left|x + 2 + \sqrt{x^2+4x+10}\right| + C$

6.2 Integrals of the Form $\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}$

By means of the inverse substitution

$$\frac{1}{mx+n} = t$$

these integrals are reduced to integrals of the form 6.1.

Example 2 Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+1}}$. Ans: $-\frac{1}{\sqrt{2}}\ln\left|\frac{1 - x + \sqrt{2\left(x^2+1\right)}}{x+1}\right|$

6.3 Integrals of the Form $\int \sqrt{ax^2+bx+c}dx$

By taking the perfect square out of the quadratic trinomial, the given integral is reduced to one of the following two basic integrals

1) $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + c; a > 0$

2) $\int \sqrt{x^2 + a^2}dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + c; a > 0$

Example 3 Evaluate $\sqrt{1-2x-x^2}dx$

7 Integration of Rational Functions

7.1 The Undetermined Coefficients

Integration of a rational function, after taking out the whole part, reduces to integration of the *proper rational fraction*

$$\frac{P(x)}{Q(x)} \qquad (1)$$

where $P(x)$ and $Q(x)$ are integral polynomials, and the degree of the numerator $P(x)$ is lower than that of the denominator $Q(x)$. If

$$Q(x) = (x-a)^{\alpha}\cdots(x-l)^{\lambda}$$

where $a, \ldots, l$ are real distinct roots of the polynomial Q(x), and $\alpha, \cdots, \lambda$ are root multiplicities, then decomposition of (1) in to partial fraction is justified:

$$\frac{P(x)}{Q(x)} \equiv \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_\alpha}{(x-a)^\alpha} + \cdots + \frac{L_1}{x-l} + \frac{L_2}{(x-l)^2} + \cdots + \frac{L_\lambda}{(x-l)^\lambda} \qquad (2)$$

where $A_1, A_2, \ldots, A_\alpha, \ldots, L_1, L_2, \ldots, L_\lambda$ are coefficients to be determined.

Example 1 Find

1) $I = \int \frac{xdx}{(x-1)(x+1)^2}$ Ans: $-\frac{1}{2(x+1)} + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + C$

2) $I = \int \frac{dx}{x^3 - 2x^2 + x}$ Ans: $\ln|x| - \ln|x-1| - \frac{1}{x-1} + C$

If the polynomial $Q(x)$ has complex roots $a \pm ib$ of multiplicity k, then partial fractions of the form

$$\frac{A_1x + B_1}{x^2 + px + q} + \cdots + \frac{A_kx + B_k}{(x^2 + px + q)^k} \qquad (3)$$

will enter into the expansion (2). Here,

$$x^2 + px + q = [x - (a+ib)][x - (a-ib)]$$

and $A_1, B_1, \ldots, A_k, B_k$ are undetermined coefficients. For k=1, the fraction (3) is integrated directly; for k>1, we use *reduction method*; here it is first advisable to represent the quadratic trinomial $x^2 + px + q$ in the form $\left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)$ and make the substitution $x + \frac{p}{2} = z$.

Example 2 Find

$$\int \frac{x+1}{(x^2 + 4x + 5)^2}dx$$

$$\text{Ans: } -\frac{x+3}{2(x^2+4x+5)} - \frac{1}{2}\tan^{-1}(x+2) + C$$

7.2 The Ostrogradsky Method

If $Q(x)$ has multiple roots, then

$$\int \frac{P(x)}{Q(x)}dx = \frac{X(x)}{Q_1(x)} + \int \frac{Y(x)}{Q_2(x)}dx \qquad (4)$$

where $Q_1(x)$ is the *greatest common divisor* of the polynomial $Q(x)$ and it derivative $Q'(x)$;

$$Q_2(x) = Q(x) : Q_1(x)$$

$X(x)$ and $Y(x)$ are polynomials with undetermined coefficients, whose degrees are, respectively, less by unity than those of $Q_1(x)$ and $Q_2(x)$.

The undetermined coefficients of the polynomials $X(x)$ and $Y(x)$ are computed by differentiating the identity (4).

Example 3 Find

$$I = \int \frac{dx}{\left(x^3-1\right)^2}$$

Solution

$$\int \frac{dx}{\left(x^3-1\right)^2} = \frac{Ax^2+Bx+C}{x^3-1} + \int \frac{Dx^2+Ex+F}{x^3-1}dx$$

Differentiating this identity, we get

$$\frac{1}{\left(x^3-1\right)^2} = \frac{(2Ax+B)\left(x^3-1\right) - 3x^2\left(Ax^2+Bx+C\right)}{\left(x^3-1\right)^2} + \frac{Dx^2+Ex+F}{x^3-1}$$

or

$$1 = (2Ax+B)\left(x^3-1\right) - 3x^2\left(Ax^2+Bx+C\right) + \left(Dx^2+Ex+F\right)\left(x^3-1\right)$$

Equating the coefficients of the respective degrees of x, we will have

$$D = 0; E - A = 0; F - 2B = 0; D + 3C = 0; E + 2A = 0; B + F = -1$$

whence

$$A = 0; B = -\frac{1}{3}; C = 0; D = 0; E = 0; F = -\frac{2}{3}$$

and, consequently,

$$\int \frac{dx}{\left(x^3-1\right)^2} = -\frac{1}{3}\frac{x}{x^3-1} - \frac{2}{3}\int \frac{dx}{x^3-1} \qquad (5)$$

To compute the integral on the right of (5), we decompose the fraction

$$\frac{1}{x^3-1} = \frac{L}{x-1} + \frac{Mx+N}{x^2+x+1}$$

we will find

$$L = \frac{1}{3}, M = -\frac{1}{3}, N = -\frac{2}{3}.$$

Therefore,

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

and

$$\int \frac{dx}{\left(x^3-1\right)^2} = -\frac{x}{3\left(x^3-1\right)} + \frac{1}{9}\ln\frac{x^2+x+1}{(x-1)^2} + \frac{2}{3\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

8 Integration of a certain Irrational functions

8.1 Integrals of the type $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}}, \ldots\right]dx$ where R is a rational function and $p_1, q_1, p_2, q_2, \ldots$ are integer numbers. We use the substitution $\frac{ax+b}{cx+d} = z^n$ where n is the *least common multiple* (lcm) of $q_1, q_2, \ldots$

Example 1 Evaluate $\int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}}$

Solution

let $2x-1 = z^4$, then $dx = 2z^3dz$, and hence

$$\int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}} = \int \frac{2z^3dz}{z^2 - z} = 2\int \frac{z^2}{z-1}dz = 2\int \left(z+1+\frac{1}{z-1}\right)dz = (z+1)^2 + 2\ln|z-1| + C$$

$$= \left(1+\sqrt[4]{2x-1}\right)^2 + \ln\left(\sqrt[4]{2x-1}-1\right)^2 + C$$

Example 2 Evaluate $\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}$ answer: $\frac{4}{3}\left[\sqrt[4]{x^3} - \ln\left(\sqrt[4]{x^3}+1\right)\right] + C$

8.2 Integrals of differential binomials $\int x^m\left(a+bx^n\right)^p dx$ where m, n and p are rational numbers.

If $\frac{m+1}{n}$ *is an integer, let* $a+bx^n = z^s$ *where s is the denominator of the fraction* $p = \frac{r}{s}$

If $\frac{m+1}{n} + p$ *is an integer, let* $ax^{-n} + b = z^s$

Example 3 Evaluate $\int \frac{x^3dx}{\left(a+bx^2\right)^{\frac{3}{2}}}$

Solution

We have $\int \frac{x^3dx}{\left(a+bx^2\right)^{\frac{3}{2}}} = \int x^3\left(a+bx^2\right)^{-\frac{3}{2}}dx$. We see that $m=3, n=2, r=-3, s=2$ and $\frac{m+1}{n} = 2$, an integer. Then assume

$a+bx^2 = z^2$, then $x = \left(\frac{z^2-a}{b}\right)^{\frac{1}{2}}$, $dx = \frac{zdz}{b^{\frac{1}{2}}\left(z^2-a\right)^{\frac{1}{2}}}$ and $\left(a+bx^2\right)^{\frac{3}{2}} = z^3$

Hence,

$$\int \frac{x^3}{\left(a+bx^2\right)^{\frac{3}{2}}}dx = \int\left(\frac{z^2-a}{b}\right)\frac{zdz}{b^{\frac{1}{2}}\left(z^2-a\right)^{\frac{1}{2}}}\frac{1}{z^3}$$

$$= \frac{1}{b^2}\int\left(1-az^{-2}\right)dz = \frac{1}{b^2}\left(z+az^{-1}\right) + C$$

$$= \frac{1}{b^2}\frac{2a+bx^2}{\sqrt{a+bx^2}} + C$$

Example 4 Work out $\int \frac{dx}{x^4\sqrt{1+x^2}} = \frac{\left(2x^2-1\right)\left(1+x^2\right)^{\frac{1}{2}}}{3x^3} + C$

8.3 Integral of the Form

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx \qquad (1)$$

where $P_n(x)$ is a polynomial of degree n

Put

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = Q_{n-1}(x)\sqrt{ax^2+bx+c}+\lambda\int\frac{dx}{\sqrt{ax^2+bx+c}} \qquad (2)$$

where $Q_{n-1}(x)$ is a polynomial of degree $(n-1)$ with undetermined coefficients are λ is number. The coefficients of the polynomial $Q_{n-1}(x)$ and the number λ are found by differentiating identity (2).

Example 5 Find $\int x^2\sqrt{x^2+4}dx$

Solution

$$\int x^2\sqrt{x^2+4}dx = \int\frac{x^4+4x^2}{\sqrt{x^2+4}}dx = \left(Ax^3+Bx^2+Cx+D\right)\sqrt{x^2+4}+\lambda\int\frac{dx}{\sqrt{x^2+4}}$$

whence

$$\frac{x^4+4x^2}{\sqrt{x^2+4}} = \left(3Ax^2+2Bx+C\right)\sqrt{x^2+4}+\frac{\left(Ax^3+Bx^2+Cx+D\right)x}{\sqrt{x^2+4}}+\frac{\lambda}{\sqrt{x^2+4}}$$

Multiplying by $\sqrt{x^2+4}$ and equating the coefficients of identical degrees of x, we obtain

$$A=\frac{1}{4}; B=0; C=\frac{1}{2}; D=0; \lambda=-2$$

Hence,

$$\int x^2\sqrt{x^2+4}dx = \frac{x^3+2x}{4}\sqrt{x^2+4}-2\ln\left(x+\sqrt{x^2+4}\right)+C$$

8.4 Integral of the form

$$\int\frac{dx}{(x-\alpha)^n\sqrt{ax^2+bx+c}} \qquad (3)$$

They are reduced to integrals of the form (1) by the substitution

$$\frac{1}{x-\alpha}=t$$

Example 6 Find $\int\frac{dx}{x^5\sqrt{x^2-1}}$

9 A Certain Trigonometric Integrals

9.1 Integral of the Form $\int\sin^n xdx$ and $\int\cos^n xdx$

If *n* is an odd positive integer, use the identity $\sin^2 x+\cos^2 x=1$

Example 1 Find $\int\sin^5 xdx$

Solution

$$\int\sin^5 xdx = \int\sin^4 x\sin xdx$$

$$= \int \left(1-\cos^2 x\right)\sin x dx$$

$$= \int \left(1-2\cos^2 x+\cos^4 x\right)\sin x dx$$

$$= -\int \left(1-2\cos^2 x+\cos^4 x\right) d\left(\cos x\right)$$

$$= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C$$

If *n* is even, use half-angled identities $\sin^2 x=\frac{1-\cos 2x}{2}$ and $\cos^2 x=\frac{1+\cos 2x}{2}$

Example 2 Find $\int \cos^4 x dx$

Solution

$$\int \cos^4 x dx=\int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$=\frac{1}{4}\int \left(1+2\cos 2x+\cos^2 2x\right)dx=\frac{1}{4}\int dx+\frac{1}{4}\int \cos 2x d\left(2x\right)+\frac{1}{8}\int \left(1+\cos 4x\right)dx$$

$$=\frac{3}{8}\int dx+\frac{1}{4}\int \cos 2x d\left(2x\right)+\frac{1}{32}\int \cos 4x d\left(4x\right)=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

Type2: $\left(\int \sin^m x\cos^n x dx\right)$

If either m or n is odd positive integer and other exponent is any number, we factor out sin*x* or cos*x* and use the identity $\sin^2 x+\cos^2 x=1$

Example 3 Find $\int \sin^3 x\cos^{-4} x dx$

Solution

$$\int \sin^3 x\cos^{-4} x dx=\int \left(1-\cos^2 x\right)\cos^{-4} x\sin x dx=-\int \left(\cos^{-4} x-\cos^{-2} x\right)d\left(\cos x\right)$$

$$=-\left[\frac{\left(\cos x\right)^{-3}}{-3}-\frac{\left(\cos x\right)^{-1}}{-1}\right]+C=\frac{1}{3}\sec^3 x-\sec x+C$$

If both *m* and *n* are even positive integers, we use half-angle identities to reduce the degree of the integrand.

Example 4 Find $\int \sin^2 x\cos^4 x dx$

Solution

$$\int \sin^2 x\cos^4 x dx=\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$=\frac{1}{8}\int \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right)dx$$

$$=\frac{1}{8}\int \left[1+\cos 2x-\frac{1}{2}\left(1+\cos 4x\right)-\left(1-\sin^2 2x\right)\cos 2x\right]dx$$

$$=\frac{1}{8}\int\left[1+\cos 2x-\frac{1}{2}(1+\cos 4x)-\left(1-\sin^2 2x\right)\cos 2x\right]dx$$

$$=\frac{1}{8}\int\left[\frac{1}{2}-\frac{1}{2}\cos 4x+\sin^2 2x\cos 2x\right]dx$$

$$=\frac{1}{8}\left[\int\frac{1}{2}dx-\frac{1}{8}\int\cos 4xd\left(4x\right)+\frac{1}{2}\int\sin^2 2xd\left(\sin 2x\right)\right]$$

$$=\frac{1}{8}\left[\frac{1}{2}x-\frac{1}{8}\sin 4x+\frac{1}{6}\sin^3 2x\right]+C$$

9.2 Integral of the Form $\int\sin mx\cos nxdx, \int\sin mx\sin nxdx, \int\cos mx\cos nxdx$

To handle these integrals, we use the *product identities*

1/. $\sin mx\cos nx=\frac{1}{2}\left[\sin\left(m+n\right)x+\sin\left(m-n\right)x\right]$

2/. $\sin mx\sin nx=-\frac{1}{2}\left[\cos\left(m+n\right)x-\cos\left(m-n\right)x\right]$

3/. $\cos mx\cos nx=\frac{1}{2}\left[\cos\left(m+n\right)x+\cos\left(m-n\right)x\right]$

Example 5 Find $\int\sin 2x\cos 3xdx$

Solution

$$\int\sin 2x\cos 3xdx=\frac{1}{2}\int\left[\sin 5x+\sin\left(-x\right)\right]=\frac{1}{10}\int\sin 5xd\left(5x\right)-\frac{1}{2}\int\sin xdx$$

$$=-\frac{1}{10}\cos 5x+\frac{1}{2}\cos x+C$$

9.3 Integrals of the Form $\int\tan^m xdx$ or $\int\cot^m xdx$ where m is a positive number

We use the formula

$$\tan^2 x=\sec^2 x-1 \text{ or } \cot^2 x=\csc^2 x-1$$

Example 6 Evaluate $\int\tan^4 xdx$

Solution

$$\int\tan^4 xdx=\int\tan^2 x\left(\sec^2 x-1\right)dx=\frac{\tan^3 x}{3}-\int\tan^2 xdx=\frac{\tan^3 x}{3}-\int\left(\sec^2 x-1\right)dx$$

$$=\frac{\tan^3 x}{3}-\tan x+x+C$$

10 Integrals of the types $\int R\left(\sin x,\cos x\right)dx$ where R is a rational function.

We can use the substitution $\tan\frac{x}{2}=t$ and hence we have

$$\sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2},\ dx=\frac{2dt}{1+t^2}$$

Example 1 Calculate $\int\frac{dx}{1+\sin x+\cos x}$

Solution

Let $\tan\frac{x}{2}=t$, then we obtain

$$I=\int\frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int\frac{dt}{1+t}=\ln|1+t|+C=\ln\left|1+\tan\frac{x}{2}\right|+C$$

If the equality $R(-\sin x,-\cos x)\equiv R(\sin x,\cos x)$ is verified, then we can make the substitution $\tan x=t$. And hence we have $\sin x=\frac{t}{\sqrt{1+t^2}}$, $\cos x=\frac{1}{\sqrt{1+t^2}}$ and $x=\arctan t,\ dx=\frac{dt}{1+t^2}$.

Example 2 Calculate $I=\int\frac{dx}{1+\sin^2 x}$

Solution

Let $\tan x=t, \sin^2 x=\frac{t^2}{1+t^2}, dx=\frac{dt}{1+t^2}$, then

$$I=\int\frac{dt}{(1+t^2)\left(1+\frac{t^2}{1+t^2}\right)}=\int\frac{dt}{1+2t^2}=\frac{1}{\sqrt{2}}\arctan\left(t\sqrt{2}\right)+C=\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right)+C$$

11 Integration of Hyperbolic Functions

Integration of hyperbolic functions is completely analogous to the integration of trigonometric function. The following basic formulas should be remembered

1) $\cosh^2 x-\sinh^2 x=1$

2) $\sinh^2 x=\frac{1}{2}(\cosh 2x-1)$

3) $\cosh^2 x=\frac{1}{2}(\cosh 2x+1)$

4) $\cosh x\sinh x=\frac{1}{2}\sinh 2x$

Example 1 Find $\int\cosh^2 xdx$

Solution

$$\int\cosh^2 xdx=\int\frac{1}{2}(\cosh 2x+1)dx=\frac{1}{4}\sinh 2x+\frac{x}{2}+C$$

Example 2 Find 1) $\int\sinh^3 x\cosh xdx$ 2) $\int\frac{\sin xdx}{\sqrt{\cosh 2x}}$ 3) $\int\sinh^2 x\cosh^2 xdx$

12 Trigonometric and Hyperbolic Substitutions for Finding Integrals of the Form

$$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx \quad (1)$$

where R is a rational function.

Transforming the quadratic trinomial ax^2+bx+c into a sum or difference of squares, the integral (1) becomes reducible to one of the following types of integrals

1) $\int R\left(z,\sqrt{m^2-z^2}\right)dz$ 2) $\int R\left(z,\sqrt{m^2+z^2}\right)dz$ $\int R\left(z,\sqrt{z^2-m^2}\right)dz$

The latter integrals are, respectively, taken by means of substitutions

1) $z=m\sin t$ or $z=m\tanh t$

2) $z = m\tan t$ or $z = m\sin t$
3) $z = m\sec t$ or $z = m\cosh t$

Example 1 find $I = \int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}$

Solution
We have $x^2+2x+2 = (x+1)^2+1$. Putting $x+1 = \tan z$, we then have $dx = \sec^2 z dz$ and

$$I = \int \frac{dx}{(x+1)^2\sqrt{(x+1)^2+1}} = \int \frac{\sec^2 z dz}{\tan^2 z \sec z} = \int \frac{\cos z}{\sin^2 z}dz = -\frac{1}{\sin z} + C = \frac{\sqrt{x^2+2x+2}}{x+1} + C$$

Example 2 Find $\int x\sqrt{x^2+x+1}dx$

Solution
We have

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

Putting

$$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\sinh t \text{ and } dx = \frac{\sqrt{3}}{2}\cosh t dt$$

we obtain

$$I = \int\left(\frac{\sqrt{3}}{2}\sinh t - \frac{1}{2}\right)\frac{\sqrt{3}}{2}\cosh t \cdot \frac{\sqrt{3}}{2}\cosh t dt = \frac{3\sqrt{3}}{8}\int \sinh t\cosh^2 t dt - \frac{3}{8}\int \cosh^2 t dt$$

$$= \frac{3\sqrt{3}}{8}\frac{\cosh^3 t}{3} - \frac{3}{8}\int \cosh^2 t dt = \frac{3\sqrt{3}}{8}\frac{\cosh^3 t}{3} - \frac{3}{8}\left(\frac{1}{2}\sinh t\cosh t + \frac{1}{2}t\right) + C$$

Since $\sinh t = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right), \cosh t = \frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$ and $t = \ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + \ln\frac{2}{\sqrt{3}}$

we finally have

$$I = \frac{1}{3}\left(x^2+x+1\right)^{\frac{3}{2}} - \frac{1}{4}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} - \frac{3}{16}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)$$

Exercises
Using basic formulas to evaluate integrals

1. $\int\left(6x^2+8x+3\right)dx$

2. $\int x(x+a)(x+b)dx$

3. $\int\left(a+bx^3\right)^2 dx$

4. $\int \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}dx$

5. $\int 3^x e^x dx$

6. $\int \frac{1-3x}{3+2x}dx$

7. $\int \sqrt{a-bx}dx$

8. $\int \frac{xdx}{2x^2+3}$

9. $\int \frac{ax+b}{a^2x^2+b^2}dx$

10. $\int \frac{x^2}{1+x^6}dx$

11. $\int \frac{x^2dx}{\sqrt{x^6-1}}$

12. $\int \sqrt{\frac{\arcsin x}{1-x^2}}dx$

13. $\int \frac{dx}{\sqrt{(1+x^2)\ln\left(x+\sqrt{1+x^2}\right)}}$

14. $\int \left(e^t - e^{-t}\right)dt$

15. $\int \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^2 dx$

16. $\int \frac{\left(a^x - b^x\right)^2}{a^x b^x} dx$

17. $\int e^{-\left(x^2+1\right)} x dx$

18. $\int x \cdot 7^{x^2} dx$

19. $\int \frac{3-2x}{5x^2+7} dx$

20. $\int \frac{3x+1}{\sqrt{5x^2+1}} dx$

21. $\int \frac{e^x}{e^x - 1} dx$

22. $\int e^x \sqrt{a - be^x} dx$

23. $\int \sin(\ln x) \frac{dx}{x}$

24. $\int \frac{\cos ax}{\sin^5 ax} dx$

25. $\int \sqrt{1+3\cos^2 x} \sin 2x dx$

26. $\int \frac{\arctan \frac{x}{2}}{4+x^2} dx$

27. $\int \frac{x - \sqrt{\arctan 2x}}{1+4x^2} dx$

28. $\int \sec^2(ax+b) dx$

29. $\int 5^{\sqrt{x}} \frac{dx}{\sqrt{x}}$

30. $\int \frac{dx}{\sin \frac{x}{a}}$

31. $\int \frac{x dx}{\cos^2 x^2}$

32. $\int x \sin\left(1-x^2\right) dx$

33. $\int \frac{dx}{\sin x \cos x} dx$

34. $\int \frac{\sin 3x}{3+\cos 3x} dx$

35. $\int \frac{\sin x \cos x}{\sqrt{\cos^2 x - \sin^2 x}} dx$

36. $\int \frac{1+\sin 3x}{\cos^2 3x} dx$

37. $\int (2\sinh 5x - \cosh 5x) dx$

38. $\int \frac{x^3 - 1}{x^4 - 4x + 1} dx$

39. $\int \frac{x^3}{x^8 + 5} dx$

40. $\int \frac{3 - \sqrt{2+3x^2}}{2+3x^2} dx$

41. $\int \frac{dx}{x \ln^2 x}$

42. $\int a^{\sin x} \cos x dx$

43. $\int \frac{x^2}{\sqrt[3]{x^3+1}} dx$

44. $\int \frac{x dx}{\sqrt{1-x^4}}$

45. $\int \frac{\sec^2 x dx}{\sqrt{4 - \tan^2 x}}$

46. $\int \frac{\sqrt[3]{1+\ln x}}{x} dx$

47. $\int \frac{e^{\arctan x} + x\ln\left(1+x^2\right) + 1}{1+x^2} dx$

48. $\int \frac{x^2}{x^2 - 2} dx$

49. $\int \frac{5-3x}{\sqrt{4-3x^2}} dx$

50. $\int \frac{dx}{e^x + 1}$

51. $\int \frac{\arccos \frac{x}{2}}{\sqrt{4-x^2}} dx$

52. Applying the indecated substitutions, find the following integrals

a) $\int \frac{dx}{x\sqrt{x^2-2}}, x = \frac{1}{t}$

b) $\int \frac{dx}{e^x+1}, x = -\ln t$

c) $\int x\left(5x^2-3\right)^7 dx, 5x^2 - 3 = t$

d) $\int \frac{x dx}{\sqrt{x+1}}, t = \sqrt{x+1}$

e) $\int \frac{\cos x dx}{\sqrt{1+\sin^2 x}}, t = \sin x$

Applying the suitable substitution, compute the following integrals

53. $\int \frac{(\arcsin x)^2}{\sqrt{1-x^2}} dx$

54. $\int x\sqrt{x^2+1} dx$

55. $\int \frac{xdx}{\sqrt{2x^2+3}}$

56. $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx$

57. $\int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{X}}}$

58. $\int \frac{dx}{\sqrt{1-x^2}\arcsin x}$

59. $\int x(2x+5)^{10} dx, t = 2x+5$

60. $\int \frac{1+x}{1+\sqrt{x}} dx, x = t^2$

61. $\int \frac{dx}{x\sqrt{2x+1}}$

62. $\int \frac{dx}{\sqrt{e^x-1}}, t^2 = e^x - 1$

63. $\int \frac{\ln 2x}{\ln 4x} \frac{dx}{x}$

64. $\int \frac{e^{2x}}{\sqrt{e^x+1}} dx$

65. $\int \frac{\sin^3 x dx}{\sqrt{\cos x}}$

66. Find the integral $\int \frac{dx}{\sqrt{x(1-x)}}$ by applying the substitution $x = \sin^2 t$

67. Find the integral $\int \sqrt{a^2+x^2} dx$ by applying the substitution $x = a\sinh t$

By using the fomula of integration by parts

68. $\int \ln x dx$

69. $\int \tan^{-1} x dx$

70. $\int \sin^{-1} x dx$

71. $\int x \sin x dx$

72. $\int x\cos 3x dx$

73. $\int \frac{x}{e^x} dx$

74. $\int x \cdot 2^{-x} dx$

75. $\int x^2 \ln x dx$

76. $\int \ln^2 x dx$

77. $\int x \tan^{-1} x dx$

78. $\int x \arcsin x dx$

79. $\int \ln\left(x+\sqrt{1+x^2}\right) dx$

80. $\int \frac{xdx}{\sin^2 x}$

81. $\int e^x \sin x dx$

82. $\int 3^x \cos x dx$

83. $\int \sin(\ln x) dx$

84. $\int (\arcsin x)^2 dx$

Integration involving quadratic trinomial expression

85. $\int \frac{dx}{2x^2-5x+7}$

86. $\int \frac{dx}{x^2+2x+5}$

87. $\int \frac{dx}{x^2+2x}$

88. $\int \frac{dx}{3x^2-x+1}$

89. $\int \frac{xdx}{x^2-7x+13}$

90. $\int \frac{3x-2}{x^2-4x+5} dx$

91. $\int \frac{x^2 dx}{x^2-6x+10}$

92. $\int \frac{dx}{\sqrt{x-x^2}} dx$

93. $\int \frac{3x-6}{\sqrt{x^2-4x+5}}dx$

94. $\int \frac{2x-8}{\sqrt{1-x-x^2}}dx$

95. $\int \frac{x}{\sqrt{5x^2-2x+1}}dx$

96. $\int \frac{dx}{x\sqrt{1-x^2}}$

97. $\int \frac{dx}{x\sqrt{x^2+x+1}}$

98. $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$

Find the Integrals

108. $\int \frac{dx}{(x+a)(x+b)}$

109. $\int \frac{x^2-5x+9}{x^2-5x+6}dx$

110. $\int \frac{dx}{(x+1)(x+2)(x+3)}$

111. $\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}dx$

112. $\int \frac{5x^3+2}{x^3-5x^2+4x}dx$

113. $\int \frac{dx}{x(x+1)^2}$

114. $\int \frac{dx}{(x^2-4x+3)(x^2+4x+5)}$

115. $\int \frac{dx}{x^3+1}$

116. $\int \frac{3x+5}{(x^2+2x+2)^2}dx$

Ostrogradsky's Method

113. $\int \frac{x^7+2}{(x^2+x+1)^2}dx$

Ans: $\frac{x}{x^2+x+1}+\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}-2\ln(x^2+x+1)+\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}+2x+C$

114. $\int \frac{(4x^2-8x)}{(x-1)^2(x^2+1)^2}dx$ **Ans:** $\frac{3x^2-x}{(x-1)(x^2+1)}+\ln\frac{(x-1)^2}{x^2+1}+\arctan x+C$

115. $\int \frac{(x^2-1)^2\,dx}{(1+x)(1+x^2)^3}$ **Ans:** $\frac{1+x}{2(1+x^2)^2}+\frac{(x-2)}{4(x^2+1)}+\frac{1}{4}\arctan x+C$

116. $\int \frac{dx}{x^4(x^3+1)^2}$ **Ans:** $\frac{2}{3}\ln\left|\frac{x^3+1}{x^3}\right|-\frac{1}{3x^3}-\frac{1}{3(x^3+1)}+C$

117. $\int \frac{dx}{(x^2+2x+10)^3}$ **Ans:** $\frac{1}{648}\left[\arctan\frac{x+1}{3}+\frac{3(x+1)}{x^2+2x+10}+\frac{18(x+1)}{(x^2+2x+10)^2}\right]+C$

118. $\int \frac{(x+2)dx}{(x^2+2x+2)^3}$ **Ans:** $\frac{3}{8}\arctan(x+1)+\frac{3}{8}\frac{x+1}{x^2+2x+2}+\frac{x}{4(x^2+2x+2)^2}+C$

119. $\int \frac{3x^4+4}{x^2(x^2+1)^3}dx$ **Ans:** $C-\frac{57x^4+103x^2+32}{8x(x^2+1)^2}-\frac{57}{8}\arctan x$

Compute integrals of the form $\int R\left[x,\left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}},\left(\frac{ax+b}{cx+d}\right)^{\frac{p_2}{q_2}},\ldots\right]dx$

120. $\int \frac{x^3}{\sqrt{x-1}}dx$

121. $\int \frac{\sqrt{x}}{x+2}dx$

122. $\int \frac{xdx}{\sqrt[3]{ax+b}}$

123. $\int \frac{dx}{(2-x)\sqrt{1-x}}$

124. $\int \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}$

125. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

126. $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx$

127. $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx$

Integration of binomial differentials

128. $\int x^3\left(1+2x^2\right)^{-\frac{3}{2}}dx$

129. $\int \frac{dx}{x\sqrt[3]{1+x^5}}$

130. $\int \frac{dx}{x^4\sqrt{1+x^2}}$

131. $\int \frac{dx}{x^2\left(2+x^3\right)^{\frac{5}{3}}}$

Trigonometric Integrals

132. $\int \cos^2 x dx$

133. $\int \sin^5 x dx$

134. $\int \sin^2 x\cos^3 x dx$

135. $\int \sin^3\frac{x}{2}\cos^5\frac{x}{2}dx$

136. $\int \sin^2 x\cos^2 x dx$

137. $\int \frac{\cos^5 x}{\sin^3 x}dx$

138. $\int \sin^3 x dx$

139. $\int \sin^2 x\cos^2 x dx$

140. $\int \sin^2 x\cos^4 x dx$

141. $\int \frac{dx}{\cos^6 x}$

142. $\int \frac{dx}{\sin^2 x\cos^4 x}$

143. $\int \frac{dx}{\sin\frac{x}{2}\cos^2\frac{x}{2}}$

144. $\int \frac{dx}{\sin^5 x}$

145. $\int \sin 3x\cos 5x dx$

146. $\int \sin 10x\sin 15x dx$

147. $\int \cos\frac{x}{2}\sin\frac{x}{2}dx$

148. $\int \sin\frac{x}{3}\sin\frac{2x}{3}dx$

149. $\int \cos(ax+b)\cos(ax-b)dx$

150. $\int \sin\omega t\sin(\omega t+\varphi)$

Integral $\int R(\sin x,\cos x)dx$

151. $\int \frac{dx}{3+5\cos x}$

156. $\int \frac{dx}{\sin x+\cos x}$

157. $\int \frac{\cos x}{1+\cos x}dx$

158. $\int \frac{dx}{8-4\sin x+7\cos x}$

159. $\int \frac{dx}{\cos x+2\sin x+3}$

160. $\int \frac{\sin x}{(1-\cos x)^3}dx$

161. $\int \frac{1+\tan x}{1-\tan x}dx$

Integrations of hyperbolic function

162. $\int \sinh^3 x dx$

163. $\int \cosh^4 x dx$

164. $\int \sinh^3 x \cosh x dx$

165. $\int \sinh^2 x \cosh^2 x dx$

166. $\int \frac{dx}{\sinh^2 x \cosh^2 x}$

167. $\int \tanh^3 x dx$

168. $\int \frac{dx}{\sinh^2 x + \cosh^2 x}$

Integral $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$

169. $\int \sqrt{3-2x-x^2}\,dx$

170. $\int \sqrt{2+x^2}\,dx$

171. $\int \sqrt{x^2-2x+2}\,dx$

172. $\int \sqrt{x^2-4}\,dx$

173. $\int \sqrt{x^2+x}\,dx$

174. $\int \sqrt{x^2-6x-7}\,dx$

175. $\int \left(x^2+x+1\right)^{\frac{3}{2}}dx$

176. $\int \frac{dx}{(x-1)\sqrt{x^2-3x+2}}$

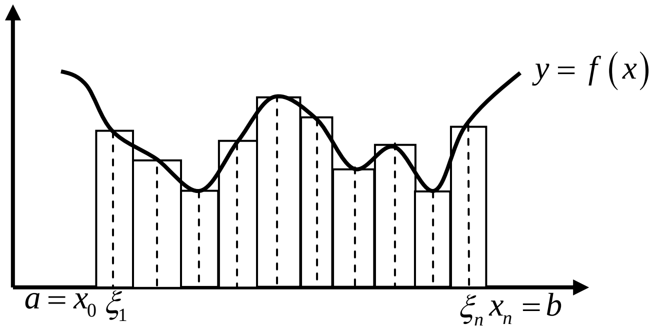
Definition Integral

1. Riemann Sum

Let $f(x)$ be a function defined over the close interval $a \le x \le b$ with $a = x_0 < x_1 < \ldots < x_n = b$ be an arbitrary partition in n subinterval. We called the *Riemann Sum* of the function $f(x)$ over $[a,b]$ the sum of the form

$$S_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

where $x_{i-1} \le \xi_i \le x_i$, $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, ..., n$.



2. Definite Integral

The limit of the sum S_n when the number of the subinterval n approaches infinity and that the largest Δx_i approaches zero is called *definite integral* of the function $f(x)$ with the upper limit $x = b$ and lower limit $x = a$.

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx$$

or equivalently

$$\lim_{n \to +\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx$$

If the function $f(x)$ is continuous on $[a,b]$ or if the limit exists, the function is said to be integrable on $[a,b]$.

If a is in the domain of f, we defined $\int_a^a f(x)\,dx = 0$ and If f is integrable on $[a,b]$, then we define $\int_b^a f(x)\,dx = -\int_a^b f(x)dx$.

Example 1 Find the Riemann Sum S_n for the function $f(x) = 1 + x$ over the interval $[1,10]$ by dividing into n equal subintervals, and then find the limit $\lim_{n\to\infty} S_n$.

Solution

$$\Delta x_i = \frac{10-1}{n} = \frac{9}{n} \qquad \xi_i = x_i = x_0 + i\Delta x_i = 1 + \frac{9i}{n}$$

and hence $f\left(\xi_i\right)=1+1+\dfrac{9i}{n}=2+\dfrac{9i}{n}$

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} f\left(\xi_i\right)\Delta x_i \\
&= \sum_{i=1}^{n}\left(2+\frac{9i}{n}\right)\frac{9}{n} \\
&= \frac{18}{n}\sum_{i=1}^{n}1+\frac{81}{n^2}\sum_{i=1}^{n}i \\
&= \frac{18}{n}n+\frac{81}{n^2}\left(1+2+\cdots+n\right) \\
&= 18+\frac{81}{n^2}\frac{n\left(n-1\right)}{2} \\
&= 18+\frac{81}{2}\left(1-\frac{1}{n}\right)
\end{aligned}$$

then

$$\lim_{n\to\infty} S_n = \lim_{x\to\infty}\left(18+\frac{81}{2}\left(1-\frac{1}{n}\right)\right)=18+\frac{81}{2}=\frac{117}{2}$$

Example 2 $\int_{-1}^{3}\left(2x^2-8\right)dx$

Solution

Divide the interval $\left[-1,3\right]$ into n equal subintervals. Hence we obtain $\Delta x_i=\dfrac{4}{n}$. In each subinterval $\left[x_{i-1},x_i\right]$, choose ξ_i such that $\xi_i=x_0+i\Delta x_i=-1+\dfrac{4i}{n}$

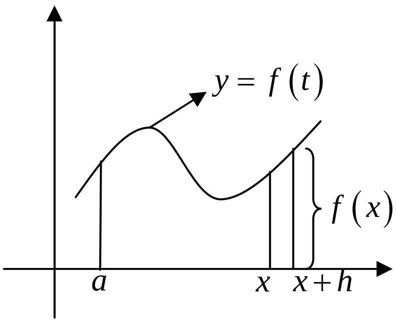
$$\begin{aligned}
\sum_{i=1}^{n} f\left(\xi_i\right)\Delta x_i &= \sum_{i=1}^{n}\left[2\left(-1+\frac{4i}{n}\right)^2-8\right]\frac{4}{n} \\
&= \sum_{i=1}^{n}\left[2\left(1-\frac{8i}{n}+\frac{16i^2}{n^2}\right)-8\right]\frac{4}{n} \\
&= \sum_{i=1}^{n}\left[-6-\frac{16i}{n}+\frac{32i^2}{n^2}\right]\frac{4}{n} \\
&= \sum_{i=1}^{n}\left(-\frac{24}{n}-\frac{64i}{n^2}+\frac{128i^2}{n^3}\right) \\
&= -\frac{24}{n}n-\frac{64}{n^2}\left(1+2+\cdots+n\right)+\frac{128}{n^3}\left(1^2+2^2+\cdots+n^2\right) \\
&= -24-\frac{64}{n^2}\cdot\frac{n\left(1+n\right)}{2}+\frac{128}{n^3}\cdot\frac{n\left(n+1\right)\left(2n+1\right)}{6} \\
&= -24-32\left(1+\frac{1}{n}\right)+\frac{128}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)
\end{aligned}$$

$$\int_{-1}^{2}\left(2x^2-8\right)dx=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\xi_i\right)\Delta x_i$$
$$=\lim_{n\to\infty}\left(-24-32\left(1+\frac{1}{n}\right)+\frac{128}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right)$$
$$=-24-32+\frac{128}{3}=-\frac{40}{3}$$

Subinterval property
If f is intergrable on an interval containing the points a, b, and c, then
$$\int_a^c f\left(x\right)dx=\int_a^b f\left(x\right)dx+\int_b^c f\left(x\right)dx$$
no matter what the order of a, b, and c.

3. The first Fundamental Theorem of Calculus

Theorem A *First Fundamental theorem of Calculus*
Let f be continuous on the closed interval $\left[a,b\right]$ and let x be a variable point in $\left(a,b\right)$, then
$$\frac{d}{dx}\int_a^x f\left(t\right)dt=f\left(x\right)$$



Proof
For $x\in\left(a,b\right)$ we define $F\left(x\right)=\int_a^x f\left(t\right)dt$, then
$$\frac{d}{dx}\int_a^x f\left(t\right)dt=F'\left(x\right)$$
$$=\lim_{h\to 0}\frac{F\left(x+h\right)-F\left(x\right)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f\left(t\right)dt-\int_a^x f\left(t\right)dt\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f\left(t\right)dt$$
But $\int_x^{x+h} f\left(t\right)dt$ represents the area bounded by x-axis the curve $f\left(t\right)$ between x and $x+h$, which is approximate to $hf\left(x\right)$; that is $\int_x^{x+h} f\left(t\right)dt\approx hf\left(x\right)$. So,
$$\frac{d}{dx}\int_a^x f\left(t\right)dt=\lim_{h\to 0}\frac{1}{h}hf\left(x\right)=f\left(x\right).$$

Example1 $\frac{d}{dx}\left[\int_2^x \frac{t^{\frac{3}{2}}}{\sqrt{t^2+17}}\,dt\right]=\frac{x^{\frac{3}{2}}}{\sqrt{x^2+7}}$

Example2

$$\frac{d}{dx}\left[\int_x^4 \tan^2 t\cos t dt\right]=\frac{d}{dx}\left[-\int_4^x \tan^2 t\cos t dt\right]$$
$$=-\frac{d}{dx}\left[\int_4^x \tan^2 t\cos t dt\right]=-\tan^2 x\cos x$$

Example3 Find $\frac{d}{dx}\left[\int_1^{x^2}(3t-1)\,dt\right]$

Solution

Let $u=x^2 \Rightarrow du=2x$ and hence

$$\frac{d}{dx}\left[\int_1^{x^2}(3t-1)\,dt\right]=\frac{d}{dx}\left[\int_1^{u}(3t-1)\,dt\right]$$
$$=\frac{d}{du}\left[\int_1^{u}(3t-1)\,dt\right]\frac{du}{dx}$$
$$=(3u-1)2x=6x^3-2x$$

Theorem B *Comparison Property*

If f and g are integrable on $[a,b]$ and if $f(x)\le g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)dx\le\int_a^b g(x)dx$$

Proof

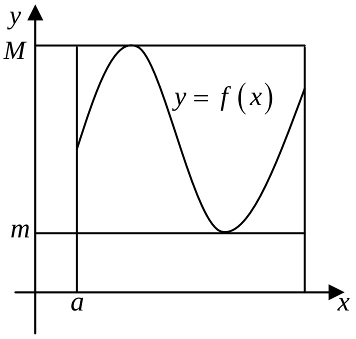
Over the interval $[a,b]$, let there be an arbitrary partition $a=x_0<x_1<\cdots<x_n=b$. Let ξ_i be a sample point on the i[th] subinterval $[x_{i-1},x_i]$, then we conclude that

$$f(\xi_i)\le g(\xi_i)$$
$$f(\xi_i)\Delta x_i\le g(\xi_i)\Delta x_i$$
$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i\le\sum_{i=1}^{n} g(\xi_i)\Delta x_i$$
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta x_i\le\lim_{n\to\infty}\sum_{i=1}^{n} g(\xi_i)\Delta x_i$$
$$\int_a^b f(x)dx\le\int_a^b g(x)dx$$

Theorem C *Boundedness Property*

If f is integrable on $[a,b]$ and $m\le f\le M$ for all x in $[a,b]$, then

$$m(b-a)\le\int_a^b f(x)dx\le M(b-a)$$



Proof

Let $h(x)=m,\forall x\in[a,b]$, then $h(x)\le f(x),\forall x\in[a,b]$.

Hence,

$$\int_a^b h(x)dx\le\int_a^b f(x)dx$$
$$m(b-a)\le\int_a^b f(x)dx$$

By similar way, let $g(x) = M, \forall x \in [a,b]$, then

$$f(x) \le g(x), \forall x \in [a,b]$$

$$\int_a^b f(x)dx \le \int_a^b g(x)\,dx$$

$$\int_a^b f(x)dx \le M(b-a)$$

Therefore $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

4. Second Fundamental Theorem of Calculus and Mean value theorem For Integrals

Second Fundamental Theorem of Calculus

Let *f* be integrable on $[a,b]$ and *F* be any primitive of *f* on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

It is also known as Newton-Leibniz Formula. For convenience we introduce a special symbol for $F(b) - F(a)$ by writing

$$F(b) - F(a) = \left[F(x)\right]_a^b \text{ or } F(b) - F(a) = F(x)\Big|_a^b$$

Example1 $\int_2^5 x^2 dx = \left.\frac{x^3}{3}\right|_2^5 = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = 39$

Example2 $\int_0^{\frac{\pi}{4}} \sin^3 2x \cos 2x dx = \left.\frac{\sin^4 2x}{8}\right|_0^{\frac{\pi}{4}} = \frac{1}{8}$

Mean Value Theorem for Integral

If *f* is continuous on $[a,b]$, there is a number *c* between *a* and *b* such that

$$\int_a^b f(t)dt = f(c)(b-a)$$

Proof

Let $F(x) = \int_a^x f(t)\,dt,\ a \le x \le b$

By Mean value theorem for derivative, we obtain

$$F(b) - F(a) = F'(c)(b-a)$$

$$\int_a^b f(t)\,dt - 0 = f(c)(b-a)$$

$$\int_a^b f(t)dt = f(c)(b-a)$$

$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$ is called the *mean value, or average value* of *f* on $[a,b]$

Example1 Find the average value of $f(x) = x^2$ on the interval $[1,4]$

Solution

$$f(x)_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 x^2 dx = \frac{1}{3}\cdot 21 = 7$$

Example2 Find the average value of $f(x)=\cos 2x$ on the interval $[0,\pi]$

5. Change of variable in definite integral

If $f(x)$ is continuous over the close interval $a \le x \le b$, if $x=\varphi(t)$ is continuous and its derivative is $\varphi'(t)$ over the interval $\alpha \le t \le \beta$, where $a=\varphi(\alpha)$ and $b=\varphi(\beta)$ and if $f[\varphi(t)]$ is defined et continuous over the interval $\alpha \le t \le \beta$, then

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,dt$$

Example1 Find $\int_0^a x^2\sqrt{a^2-x^2}\,dx \qquad (a>0)$

Solution

Let $x=a\sin t, dx=a\cos t$, $t=\arcsin\frac{x}{a}$, $\alpha=\arcsin 0=0$ and $\beta=\arcsin 1=\frac{\pi}{2}$. then we obtain

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx=\int_0^{\frac{\pi}{2}} a^2\sin^2 t\left(\sqrt{a^2-a^2\sin^2 t}\right)a\cos t dt$$

$$=a^4\int_0^{\frac{\pi}{2}}\sin^2 t\cos^2 t dt=\frac{a^4}{4}\int_0^{\frac{\pi}{2}}\sin^2 2t dt$$

$$=\frac{a^4}{8}\int_0^{\frac{\pi}{2}}(1-\cos 4t)dt=\frac{a^4}{8}\left(1-\frac{1}{4}\sin 4t\right)\Big|_0^{\frac{\pi}{2}}=\frac{\pi a^4}{16}$$

Example2 Evaluate $\int_0^4\frac{dx}{1+\sqrt{x}}$ let $x=t^2$ (answer: $4-2\ln 3$)

Example3 Evaluate $\int_0^{\ln 2}\sqrt{e^x-1}dx$ let $e^x-1=z^2$ (answer: $2-\frac{\pi}{2}$)

6. Integration by parts

If the functions $u(x)$ and $v(x)$ are continuous differentiable over $[a,b]$, we have

$$\int_a^b u(x)v'(x)\,dx=u(x)v(x)\Big|_a^b-\int_a^b v(x)u'(x)\,dx$$

Example1 Evaluate $\int_0^{\frac{\pi}{2}} x\cos x dx$ (answer: $\frac{\pi}{2}-1$)

Example2 Evaluate $\int_0^1 x^3e^{2x}dx$ (answer: $\frac{e^2+3}{8}$)

Example3 Evaluate $\int_0^\pi e^x\sin x dx$ (answer: $\frac{1}{2}\left(e^\pi+1\right)$)

7. Improper Integral

Improper integrals refer to those involving in the case where the interval of integration is infinite and also in the case where *f* (the integrand) is unbounded at a finite number of points on the interval of integration.

7.1 Improper Integral with Infinite Limits of Integration

Let *a* be a fixed number and assume that $\int_a^N f(x)dx$ exists for all $N \ge a$. Then if

$\lim\limits_{N\to+\infty}\int_a^N f(x)\,dx$ exists, we define the improper integral $\int_a^{+\infty} f(x)\,dx$ by

$$\int_a^{+\infty} f(x)dx = \lim_{N\to+\infty}\int_a^N f(x)dx$$

The improper integral is said to be *convergent* if this limit is a finite number and to be *divergent* otherwise.

Example Evaluate $I = \int_1^{+\infty}\frac{dx}{x^2}$

Solution

$$\int_1^{+\infty}\frac{dx}{x^2} = \lim_{N\to+\infty}\int_1^N\frac{dx}{x^2} = \lim_{N\to+\infty}\left(-\frac{1}{x}\right)\Bigg|_1^N = \lim_{N\to+\infty}\left(-\frac{1}{N}+1\right)=1$$

Thus, the improper integral converges and has the value 1.

Example Evaluate $\int_1^{+\infty}\frac{dx}{x^p}$ $\int_0^{+\infty} xe^{-2x}dx$

Let *b* be a fixed number and assume $\int_t^b f(x)\,dx$ exists for all $t < b$. Then if

$\lim\limits_{t\to-\infty}\int_t^b f(x)\,dx$ exists we define the improper integral

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

The improper integral $\int_{-\infty}^b f(x)\,dx$ is said to be converge if this limit is a finite number and to diverge otherwise. If both $\int_a^{+\infty} f(x)\,dx$ And $\int_{-\infty}^a f(x)\,dx$ converge for some number *a*, the improper integral of $f(x)$ on the entire x-axis is defined by

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^a f(x)dx + \int_a^{+\infty} f(x)\,dx$$

Example Evaluate $\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}$ (answer: π) $\int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}$ (answer: π)

7.2 Improper Integrals with Unbounded Integrands

If *f* is unbounded at *a* and $\int_t^b f(x)\,dx$ exists for all *t* such that $a < t \le b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

If the limit exists (as a finite number), we say that the improper integral converge; otherwise, the improper integral diverges. Similarly, if *f* is unbounded at *b* and $\int_a^t f(x)\,dx$ exists for all *t* such that $a \le t < b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

If *f* is unbounded at *c* where $a < c < b$ the improper integral $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ both converge, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

We say that the integral on the left diverges if either or both of the integrals on the right diverge.

Example Find $\int_0^1 \frac{dx}{(x-1)^{2/3}}$ $\int_0^3 \frac{dx}{x-2}$

Note:

1. For $x \ge a$, if $0 \le f(x) \le g(x)$ and if $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converge and $\int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$

Example Investigate the convergence of $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$

2. For $x \ge a$, if $0 \le f(x) \le g(x)$ and if $\int_a^{+\infty} f(x)\,dx$ diverges, then $\int_a^{+\infty} g(x)\,dx$ diverges.

Example Investigate the convergence of $\int_1^{+\infty} \frac{x+1}{\sqrt{x^3}}dx$

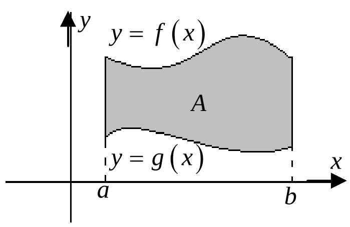
3. If $\int_a^{+\infty} |f(x)|\,dx$ is convergent then $\int_a^{+\infty} f(x)\,dx$ is also convergent, specifically *absolute convergent.*

Example Investigate the convergence of $\int_1^{+\infty} \frac{\sin x}{x^3}dx$

8 Area Between Two Curves

8.1 Area Between $y = f(x)$ and $y = g(x)$

If f and g are continuous functions on the interval $[a,b]$, and if $f(x) \ge g(x)$ for all *x* in $[a,b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by line $x = a$, and on the right by the line $x = b$ is defined by



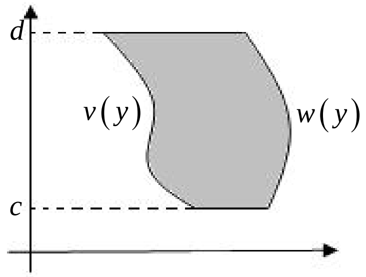
$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

Example1 Find the area of region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$. ans: $\frac{34}{3}$

Example2 Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$. $\frac{125}{6}$

8.2 Area Between $x = v(y)$ and $x = w(y)$

If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is defined by

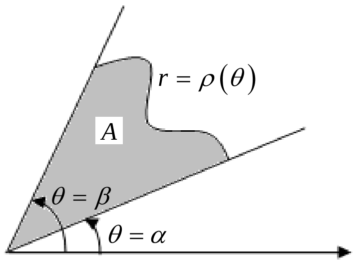


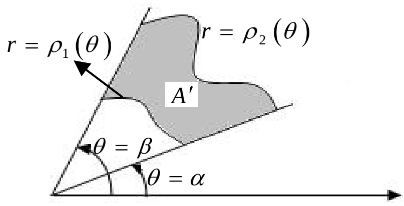
$$A = \int_c^d \left[w(y) - v(y) \right] dy$$

Example1 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y. (ans: $\frac{9}{2}$)

Example2 Find the area of the region enclosed by the curves $y = x^2$ and $y = 4x$ by integrating *a/.* with respect to x *b/.* with respect to y

8.3 Area in Polar Coordinates





$$A = \frac{1}{2} \int_\alpha^\beta \left[\rho(\theta) \right]^2 d\theta \qquad A' = \frac{1}{2} \int_\alpha^\beta \left(\left[\rho_2(\theta) \right]^2 - \left[\rho_1(\theta) \right]^2 \right) d\theta$$

Example Calculate the area enclosed by the cardioid $r = 1 - \cos\theta$ (answer: $\frac{3\pi}{2}$)

Example Find the area of region that is inside the cardioid $r = 4 + 4\cos\theta$ and outside the circle $r = 6$ (answer: $18\sqrt{3} - 4\pi$).

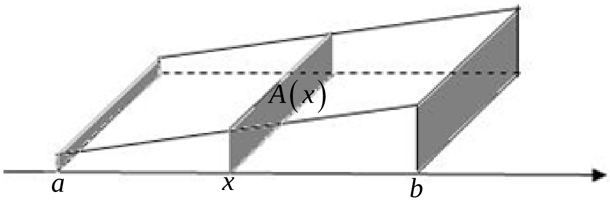
9 Volume of Solid

9.1 Volume By Cross Sections Perpendicular To The *X*-Axis

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in the interval $[a, b]$, the cross-sectional area of S

perpendicular to the *x*-axis is $A(x)$, then the volume of the solid, provided $A(x)$ is integrable, is defined by

$$V = \int_a^b A(x)dx$$



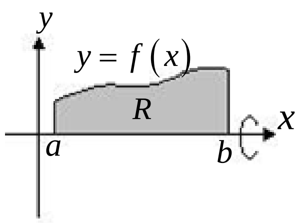
9.2 Volume By Cross Sections Perpendicular To The *Y*-Axis

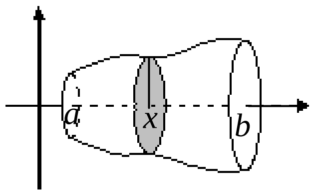
S be a solid bounded by two parallel planes perpendicular to the *y*-axis at $y = c$ and $y = d$. If, for each *y* in the interval $[c, d]$, the cross-sectional area of S perpendicular to the *y*-axis is $A(y)$, then the volume of the solid, provided $A(y)$ is integrable, is defined by

$$V = \int_c^d A(y)dy$$

Example 1 Derive the formula for the volume of a right pyramid whose altitude is *h* and whose base is a square with sides of length *a*. $\frac{1}{3}a^2h$.

9.3 Volumes of Solids Of Revolution





2.3.a Volumes by Disks Perpendicular To the *x*-Axis

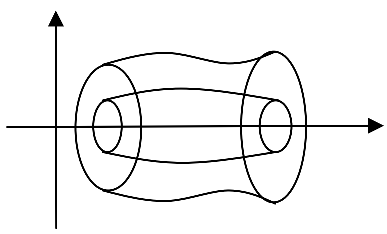
$$V = \int_a^b \pi\left[f(x)\right]^2 dx$$

Example 2 Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the *x*-axis.(ans: $\frac{15\pi}{2}$)

Example 3 Derive the formula for the volume of a sphere of radius *r*. (ans: $\frac{4}{3}\pi r^3$)

2.3.b Volumes by Washers Perpendicular to the *x*-Axis

Suppose that f and g are nonnegative continuous functions such that $g(x) \le f(x)$ for $a \le x \le b$. Let R be the region enclosed between the graphs of these functions and lines $x = a$ and $x = b$. When this region is revolved about the x-axis, it generates a solid whose volumes is defined by



$$V = \int_a^b \pi\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$$

Example 4 Find the volume of the solid generated when the region between the graphs of $f(x)=\frac{1}{2}+x^2$ and $g(x)=x$ over the interval $[0,2]$ is revolved about the *x-axis.* Ans: $\frac{69\pi}{10}$

2.3.*c *Volumes By Disks Perpendicular To the y-axis

$$V=\int_c^d \pi\left[u(x)\right]^2 dy$$

***2.3.*d Volumes By Washers Perpendicular To y-axis**

$$V=\int_c^d \pi\left(\left[u(y)\right]^2-\left[v(y)\right]^2\right)dy$$

2.3.e Cylindrical Shells Centered on the y-axis

Let *R* be the a plane region bounded above by a continuous curve $y=f(x)$, below by the *x*-axis, and on the left and right respectively by the lines $x=a$ and $x=b$. Then the volume of the solid generated by revolving *R* about the y-axis is given by

$$V=2\pi\int_a^b xf(x)\,dx$$

Example 5 Find the volume of the solid generated when the region enclosed between $y=\sqrt{x}$, $x=1, x=4$ and the x-axis revolved about the y-axis.

Solution

Since $f(x)=\sqrt{x}, a=1, b=4$, then the volume of the solid is

$$V=2\pi\int_1^4 \sqrt{x}dx=2\pi\int_1^4 x^{3/2}dx=2\pi\cdot\frac{2}{5}x^{\frac{5}{2}}\Big|_1^4=\frac{4\pi}{5}[32-1]=\frac{124\pi}{5}$$

Example 6 Find the volume of the solid generated when the region *R* in the first quadrant enclosed between $y=x$ and $y=x^2$ is revolved about the y-axis. (Answer: $\pi/6$)

3 Length of a Plane Curve

If f is a smooth function on $[a,b]$, then the *arc length* ***L*** of the curve $y=f(x)$ $x=a$ to $x=b$ is defined by

$$L=\int_a^b\sqrt{1+\left[f'(x)\right]^2}dx=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

Similarly, for a curve expressed in the form $x=g(y)$ where g' is continuous on $[c,d]$, the arc length *L* from $y=c$ to $y=d$ defined by

$$L=\int_c^d\sqrt{1+\left[g'(y)\right]^2}dy=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$$

Example 1 Find the arc length of $f(x)=x^2$ from $(0,0)$ to $(1,1)$

Solution

$$f(x)=x^2 \Rightarrow f'(x)=2x$$

$$\sqrt{1+[f'(x)]^2}=\sqrt{1+4x^2}=\frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2+x^2}$$

Then the arc length is defined by

$$L=\frac{1}{2}\int_0^1\sqrt{\left(\frac{1}{2}\right)^2+x^2dx}$$

$$=\left[x\sqrt{\left(\frac{1}{2}\right)^2+x^2}+\left(\frac{1}{2}\right)^2\ln\left(x+\sqrt{x^2+\left(\frac{1}{2}\right)^2}\right)\right]_0^1$$

$$=\frac{1}{2}\sqrt{5}+\frac{1}{4}\ln\left(2+\sqrt{5}\right)$$

If the curve is given in polar coordinate system $r=\rho(\theta),\alpha\le\theta\le\beta$ then the arc length of the curve is defined by

$$L=\int_\alpha^\beta\sqrt{[\rho(\theta)]^2+[\rho'(\theta)]^2}d\theta=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta$$

Example 2 Find the circumference of the circle or radius *a*.

Solution

As a polar equation this circle is denoted by $r=a$, $0\le\theta\le2\pi$

Then the arc length is $L=\int_0^{2\pi}\sqrt{a^2}d\theta=a\int_0^{2\pi}d\theta=a\theta\Big|_0^{2\pi}=2\pi a$

Example 3 Find the length of the cardioid $r=1-\cos\theta$

If the curve is defined by the parametric equation $x=x(t),y=y(t),t\in[a,b]$, then the length of the curve is

$$L=\int_a^b\sqrt{[x'(t)]^2+[y'(t)]^2}dt$$

Example 4 Find the circumference of the circle of the radius *r*

Solution

Parametric form, the circle is defined by $x(t)=r\cos t, y(t)=r\sin t$ with $t\in[0,2\pi]$, then

$$L=\int_0^{2\pi}\sqrt{r^2\cos^2 t+r^2\sin^2 t}dt=\int_0^{2\pi}rdt=2\pi r$$

Example 5 Find the arc length of the astroid $x(t)=a\cos^3 t, y(t)=a\sin^3 t$.(ans6*a*).

4 Area of Surface of Revolution

Let *f* be a smooth, nonnegative function on $[a,b]$. Then the surface area *S* generated by revolving the portion of the curve $y=f(x)$ between $x=a$ and $x=b$ about x-axis is

$$S = 2\pi\int_a^b f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

For a curve expressed in the form $x = g(y)$ where g' is continuous on $[a,d]$ and $g(y) \ge 0$ for $c \le y \le d$, the surface area S generated by revolving the portion of the curve from $y = c$ to $y = d$ about the y-axis is given by

$$S = 2\pi\int_c^d g(y)\sqrt{1+\left[g'(y)\right]^2}\,dy$$

Example1 Find the surface area generated by revolving the curve $y = \sqrt{1-x^2}$, $0 \le x \le \frac{1}{2}$ about the x-axis.

Solution

$f(x) = \sqrt{1-x^2} \Rightarrow f'(x) = \dfrac{-x}{\sqrt{1-x^2}}$ **.** Thus,

$$S = 2\pi\int_0^{1/2}\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx = 2\pi\int_0^{1/2} dx = \pi$$

Example2 Find the surface area generated by revolving the curve $y = \sqrt[3]{3x}, 0 \le y \le 2$ about the y-axis.

Solution

$y = \sqrt[3]{3x} \Rightarrow x = g(y) = \dfrac{1}{3}y^3$. Thus, $g'(y) = y^2$, then

$$S = 2\pi\int_0^2\left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\,dy = \frac{2\pi}{3}\int_0^2 y^3\sqrt{1+y^4}\,dy$$

$$= \frac{2\pi}{3}\left[\frac{1}{6}\left(1+y^4\right)^{3/2}\right]_0^2 = \frac{\pi}{9}\left(17^{3/2}-1\right)$$

Exercises

Work out the following integrals

1. $\displaystyle\int_0^2 \frac{x^3dx}{x+1} = \frac{8}{3} - \ln 3$

2. $\displaystyle\int_0^{16} \frac{x^{\frac{1}{4}}}{1+x^{\frac{1}{4}}}dx = \frac{8}{3} + 4\arctan 2$

3. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x\cos^3 x dx = \frac{1}{12}$

4. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4\theta d\theta = \frac{4}{3}$

5. $\displaystyle\int_3^{29} \frac{(x-2)^{\frac{2}{3}}\,dx}{(x-2)^{\frac{2}{3}}+3} = 8 + \frac{3\sqrt{3}\pi}{2}$

6. $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \frac{\pi}{3\sqrt{3}}$

7. $\int_1^2 \frac{x-3}{x^3+x^2}dx = 4\ln\frac{4}{3}-\frac{3}{2}$

8. $\int_0^1 \frac{dx}{e^x+e^{-x}} = \arctan e-\frac{\pi}{4}$

9. $\int_0^{\frac{\pi}{2}} \sin^4 xdx = \frac{3\pi}{16}$

10. $\int_0^{\pi} \cos^4 xdx = \frac{3\pi}{8}$

11. $\int_1^{e^2} \frac{dx}{x(1+\ln x)} = \ln 3$

12. $\int_1^{e^2} \frac{dx}{x(1+\ln x)^2} = \frac{2}{3}$

13. $\int_0^1 \frac{xdx}{x^2+3x+2} = \ln\frac{9}{8}$

14. $\int_0^1 \frac{z^3}{z^8+1}dz = \frac{\pi}{16}$

15. $\int_0^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{4}$

16. $\int_2^5 \frac{dx}{\sqrt{5+4x-x^2}} = \frac{\pi}{2}$

17. $\int_0^{\frac{\pi}{2}} \sin^3 xdx = \frac{2}{3}$

18. $\int_e^{e^2} \frac{dx}{x\ln x} = \ln 2$

Find the derivative of the following functions

19. $F(x) = \int_1^x \ln tdt$**,** Ans: ln$x$

20. $\int_x^0 \sqrt{1+t^4}dt$**,** Ans: $-\sqrt{1+x^4}$

21. $F(x) = \int_x^{x^2} e^{-t^2}dt$**,** Ans: $-e^{-x^2}+2xe^{-x^4}$

22. $F(x) = \int_{\frac{1}{x}}^{\sqrt{x}} \cos(t^2)dt$**,** Ans: $\frac{1}{x^2}\cos\left(\frac{1}{x^2}\right)+\frac{1}{2\sqrt{x}}\cos x$

Work out the following integrals

23. $\int_0^1 \frac{xdx}{\sqrt{1-x^2}} = 1$

24. $\int_0^{\infty} e^{-x}dx = 1$

25. $\int_0^{+\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{2a}, (a>0)$

26. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$

27. $\int_0^1 \ln x dx = -1$

28. $\int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2} = \pi$

29. $\int_0^9 \frac{dx}{(x-1)^{2/3}} = 9$

30. $\int_e^\infty \frac{dx}{x\ln x\sqrt{\ln x}} = 2$

31. $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2\sqrt{a}}$

32. $\int_0^\infty \frac{dx}{e^x+e^{-x}} = \frac{\pi}{4}$

33. $\int_2^\infty \frac{dx}{x\ln^2 x} = \frac{1}{\ln 2}$

Compute the improper integrals (or prove their divergence)

34. $\int_1^\infty \frac{dx}{x^4}$

35. $\int_0^\infty e^{-ax} dx, a > 0$

36. $\int_{-\infty}^{+\infty} \frac{2xdx}{x^2+1}$

37. $\int_2^{+\infty} \frac{\ln x}{x} dx$

38. $\int_1^\infty \frac{dx}{x^2(x+1)}$

39. $\int_0^\infty \frac{dx}{(1+x)^3}$

40. $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

41. $\int_{a^2}^\infty \frac{dx}{x\sqrt{1+x^2}}$

42. $\int_0^\infty xe^{-x^2} dx$

43. $\int_0^\infty x^3 e^{-x^2} dx$

44. $\int_1^\infty \frac{\arctan x}{x^2} dx$

45. $\int_0^\infty \frac{dx}{1+x^3}$

46. $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2}$

47. For $p \le 1$**,** is $\int_1^\infty \frac{\ln x}{x^p} dx$ convergent? (Hint: $\frac{\ln x}{x^p} \ge \frac{1}{x^p}$ for $x \ge e$)

48. For what values of *k* are the integrals $\int_2^\infty \frac{dx}{x^k \ln x}$ and $\int_2^\infty \frac{dx}{x(\ln x)^k}$ convergent?

49. For what values of *k* is the integral $\int_a^b \frac{dx}{(b-x)^k}, (b<a)$ convergent?

50. Show that $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even and $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is odd.

51. Show that $\int\limits_{-\infty}^{\infty} e^{-x^2} dx = 2\int\limits_0^\infty e^{-x^2} dx = \int\limits_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$

52. Show that $\int\limits_0^1 \frac{dx}{\arccos x} = \int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx$

53. $\int\limits_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int\limits_0^{\frac{\pi}{2}} f(\cos x)\,dx$

54. The **Laplace Transformation** of the function *f* is defined by the improper integral

$$F(s) = \mathscr{L}\{f(t)\} = \int\limits_0^{+\infty} e^{-st} f(t)\,dt.$$

Show that for constant *a* (with $s - a > 0$)

a. $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$ **b.** $\mathscr{L}\{a\} = \frac{a}{s}$ **c.** $\mathscr{L}\{t\} = \frac{1}{s^2}$ **d.** $\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2}$

e. $\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}$

55. Find the first quadrant area under the curve $y = e^{-2x}$ (answer: $\frac{1}{2}$)

56. Let $\mathscr{R}$ be the region in the first quadrant under $xy = 9$ and to the right of $x = 1$. Find the volume generated by revolving $\mathscr{R}$ about the x-axis. (answer: 81π)

57. Derive a formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a right circular cone of height *h* and radius of base *r*.

58. Let $\mathscr{R}$ be the region above the curve $y = x^3$ under the line $y = 1$ and between $x = 0$ and $x = 1$. Find the volume generated by revolving $\mathscr{R}$ about a). x-axis, b). about y-axis.

(answer: a). $\frac{6}{7}\pi$, b). $\frac{3}{5}\pi$)

59. Find the area of the region between $y = x^3$ and the lines $y = -x$ and $y = 1$

60. Find the area of the region bounded by the curve $y = \sin x$, $y = \cos x$ and $x = 0$ and $x = \pi/4$ (answer: $\sqrt{2} - 1$)

61. Find the area of the region bounded by parabolas $y = x^2$ and $y = -x^2 + 6x$.
(Answer: 9)

62. Find the area of the region bounded by the parabola $x = y^2 + 2$ and the line $y = x - 8$. (answer: $\frac{125}{6}$)

62. Find the area of the region bounded by the parabolas $y = x^2 - x$ and $y = x - x^2$.

(Answer: $\frac{1}{3}$)

62. Find the arc length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$ (ans: $\frac{33}{16}$)

63. Find the arc length of the curve $x^{2/3} + y^{2/3} = 4$ from $x = 1$ to $x = 8$ (ans: 9)

64. Find the arc length of the curve $6xy = x^4 + 3$ from $x = 1$ to $x = 2$ (ans: $\frac{17}{12}$)

65. Find the area inside the cardioid $r = 1 + \cos\theta$ and outside $r = 1$ (ans: $\frac{\pi}{4} + 2$)

66. Find the area inside the circle $r = \sin\theta$ and outside the cardioid $r = 1 - \cos\theta$

67. Find the volume generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x-axis.

Answer: $\frac{4}{3}\pi ab^2$

Infinite Series

1. SEQUENCES AND THEIR LIMITS

Sequences

A sequence $\{a_n\}$ is a function whose domain is a set of nonnegative integers and whose range is the subset of real number. The functional value $a_1, a_2, a_3 \ldots$ are called **terms** of the sequence and a_n is called the **nth term,** or **general term** of the sequence.

Limit of the sequence

If the terms of the sequence approach the number L as n increases without bound, we say that the sequence converges to the limit L and write

$$L = \lim_{n\to+\infty} a_n$$

Convergent sequence

The sequence $\{a_n\}$ converges to the number L, and we write $L = \lim_{n\to\infty} a_n$ if for every $\varepsilon > 0$, there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Otherwise, the sequence diverges.

Limit Theorem for Sequences

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. Linearity Rule: $\lim_{n\to\infty}(ra_n + sb_n) = rL + sM$
2. Product Rule: $\lim_{n\to\infty}(a_n b_n) = LM$
3. Quotient Rule: $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{L}{M}$ provided $M \neq 0$
4. Root Rule: $\lim_{n\to\infty}\sqrt[m]{a_n} = \sqrt[m]{L}$ provided $\sqrt[m]{a_n}$ is defined for all n and $\sqrt[m]{L}$ exists.

Example:

Find the limit of the convergent sequences

a/. $\left\{\dfrac{2n^2+5n-7}{n^3}\right\}$ b/. $\left\{\dfrac{3n^4+n-1}{5n^4+2n^2+1}\right\}$ c/. $\left\{\sqrt{n^2+3n}-n\right\}$

Limit of a sequence from the limit of a continuous function

The sequence $\{a_n\}$, let f be a continuous function such that $a_n = f(n)$ for $n = 1, 2, 3, \ldots$ If $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) = L$, the sequence $\{a_n\}$ converge and $\lim_{n\to\infty} a_n = L$.

Example: Given that the $\left\{\dfrac{n^2}{1-e^n}\right\}$ converges, evaluate $\lim_{n\to\infty}\dfrac{n^2}{1-e^n}$

Bounded, Monotonic Sequences

Name	**Condition**
Strictly increasing	$a_1 < a_2 < \ldots < a_{k-1} < a_k < \ldots$
Increasing	$a_1 \le a_2 \le \ldots \le a_{k-1} \le a_k \le \ldots$
Strictly decreasing	$a_1 > a_2 > \ldots > a_{k-1} > a_k > \ldots$
Decreasing	$a_1 \ge a_2 \ge \ldots \ge a_{k-1} \ge a_k \ge \ldots$
Bounded above by M	$a_n \le M$ for $n = 1, 2, 3, \ldots$
Bounded below by m	$m \le a_n$ for $n = 1, 2, 3, \ldots$
Bounded	If it is bounded both above and below

2. INFINITE SERIES; GEOMETRIC SERIES

An **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

and the **nth partial sum of** the series is

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

The series is said to **converge with sum** S if the sequence of partial sums $\{s_n\}$ converges to S. In this case, we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S$$

If the sequence $\{S_n\}$ does not converge, the series $\sum_{k=1}^{\infty} a_k$ diverges and has no sum.

Example: Show that the series $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$ converges and find its sum.

Solution:

We have $\frac{1}{k^2+k} = \frac{1}{k} - \frac{1}{k+1}$. Then

$$S_n = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$$

Example: Prove that the series convergent and find its sum

a. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ **c.** $\sum_{k=1}^{\infty} \frac{1}{2^k}$

Geometric Series

A geometric series is an infinite series in which the ratio of successive term in the series is constant. If this constant ratio is r, then the series has the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots, a \ne 0$$

Geometric Theorem

The geometric series $\sum_{k=0}^{\infty} ar^k$ with $a \neq 0$ diverges if $|r| \geq 1$ and converges if $|r| < 1$ with sum $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

Proof:

The nth partial sum of the geometric series is $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$.

Then, $rS_n = ra + ar^2 + ar^3 + \cdots + ar^n$

$$\Rightarrow rS_n - S_n = ar^n - a$$

$$\Rightarrow S_n = \frac{a(r^n - 1)}{r-1}, r \neq 1$$

If $|r| > 1 \Rightarrow r^n \xrightarrow{n\to\infty} \infty \Rightarrow \lim_{n\to\infty} S_n = \infty$

If $|r| < 1 \Rightarrow r^n \xrightarrow{n\to\infty} 0 \Rightarrow \lim_{n\to\infty} S_n = \frac{a}{1-r}$

THE INTEGRAL TEST, p-series

Divergent Test

If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum a_k$ must diverge.

Proof:

Suppose the sequence of partial sums $\{S_n\}$ converges with sum *L*, so that $\lim_{n\to\infty} S_n = L$. Then we also have $\lim_{n\to\infty} S_{n-1} = L$.

We have $S_k - S_{k-1} = a_k$, and then it follows that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = L - L = 0$$

We see that if $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Thus, if $\lim_{k\to\infty} a_k \neq 0$, then $\sum a_k$ diverges.

Example:

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k}{k+1} + \cdots \text{ Diverges since}$$

$$\lim_{k\to\infty} \frac{k}{k+1} = 1 (\neq 0)$$

The Integral Test

If $a_k = f(k)$ for $k = 1, 2, 3, \ldots$ where *f* is a positive continuous and decreasing function of *x* for $x \geq 1$ then

$$\sum_{k=1}^{\infty} a_k \text{ And } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Example: Test the series $\sum_{k=1}^{\infty} \frac{1}{k}$ for convergence

Solution:

We have $f(x) = \frac{1}{x}$ is a positive, continuous and decreasing for $x \geq 1$.

$\int_1^{\infty} \frac{1}{x}dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}dx = \lim_{b\to\infty} [\ln b] = \infty$, implying that $\int_1^{\infty} \frac{1}{x}dx$ diverges.

Hence $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

Example: Investigate the following series for convergent

1. $\sum_{k=1}^{\infty}\frac{k}{e^{k/5}}$ 2. $\sum_{k=1}^{\infty}\frac{1}{k^2}$ 3. $\frac{1}{e}+\frac{2}{e^4}+\cdots+\frac{k}{e^{k^2}}+\cdots$

p-series

A series of the form

$$\sum_{k=1}^{\infty}\frac{1}{k^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

where p is a positive constant, is called **a p-series.**

Note: The harmonic series is the case where $p=1$.

Theorem, the p-series test

The p-series $\sum_{k=1}^{\infty}\frac{1}{k^p}$ converges if $p>1$ and diverges if $p\le 1$.

Proof:

Let $f(x)=\frac{1}{x^p}$ $f'(x)=-\frac{px^{p-1}}{x^{2p}}$ then $f'(x)<0$ if $p>0$

Hence $f(x)=\frac{1}{x^p}$ is continuous, positive and decreasing $x\ge 1$ and $p>0$.

For $p=1$, the series is harmonic, that is it diverges

For $p>0$ and $p\ne 1$ we have:

$$\int_1^{\infty}\frac{dx}{x^p}=\lim_{b\to\infty}\int_1^{b}x^{-p}dx=\lim_{b\to\infty}\frac{b^{1-p}-1}{1-p}=\begin{cases}\frac{1}{p-1}, p>1\\ \infty, 0<p<1\end{cases}$$

That is, this improper integral converges if $p>1$ and diverges if $0<p<1$

For $p=0$, the series becomes

$$\sum_{k=1}^{\infty}\frac{1}{k^0}=\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\cdots$$

For $p<0$, we have $\lim_{k\to\infty}\frac{1}{k^p}=\infty$, so the series diverges by the convergence test.

Hence, a p-series converges only when $p>1$.

Example: Test each of the following series for convergence

a. $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k^3}}$ **b.** $\sum_{k=1}^{\infty}\left(\frac{1}{e^k}-\frac{1}{\sqrt{k}}\right)$

Solution:

a. $\sqrt{k^3}=k^{3/2}$. So $p=3/2>1$ and the series converges.

b. We have $\sum_{k=1}^{\infty}\frac{1}{e^k}$ converges, because it is a geometric series with $|r|=\frac{1}{e}<1$.

And $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ diverges because it is a p-series with $p=\frac{1}{2}<1$

Hence $\sum_{k=1}^{\infty}\left(\frac{1}{e^k}-\frac{1}{\sqrt{k}}\right)$ diverges.

4. COMPARISON TEST

❶ Direct Comparison Test

Suppose $0 \le a_k \le c_k$ for all $k \ge N$ for some N. If $\sum_{k=1}^{\infty} c_k$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

Let $0 \le d_k \le a_k$ for all $k \ge N$ for some *N***.** If $\sum_{k=1}^{\infty} d_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ also diverges.

Example: Test the series $\sum_{k=1}^{\infty} \frac{1}{3^k+1}$ for convergence.

Solution:

We have $3^k+1 > 3^k > 0$ for $k \ge 1$. Then $0 < \frac{1}{3^k+1} < \frac{1}{3^k}$. Since $\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges, it implies that $\sum_{k=1}^{\infty} \frac{1}{3^k+1}$ converges.

Example: Test for convergence the following series

a. $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}-1}$ **b.** $\sum_{k=1}^{\infty} \frac{1}{k!}$

❷ Limit Comparison Test

Suppose $a_k > 0$ and $b_k > 0$ for all sufficiently large *k* and that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ where *L* is finite and positive $(0 < L < \infty)$. Then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

Example: Test the series $\sum_{k=1}^{\infty} \frac{1}{2^k-5}$ for convergence.

Solution:

We see that $\sum \frac{1}{2^k}$ is a convergent series for it is the geometric series with $|r| = \frac{1}{2} < 1$. Moreover

$$\lim_{k\to\infty} \frac{\frac{1}{2^k-5}}{\frac{1}{2^k}} = \frac{2^k}{2^k-5} = 1$$

Hence $\sum \frac{1}{2^k-5}$ is convergent too.

The zero-infinity limit comparison test

Suppose $a_k > 0$ and $b_k > 0$ for all sufficient large *k*.

If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges

If $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

5. THE RATIO TEST AND THE ROOT TEST

Theorem: Given the series $\sum a_k$ with $a_k > 0$, suppose that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$

The ratio test states the following:

If $L<1$, then $\sum a_k$ converges

If $L>1$, then $\sum a_k$ diverges

If $L=1$, then the test is inconclusive

Example: Test the series $\sum_{k=1}^{\infty}\frac{2^k}{k!}$ for convergence.

Solution:

Let $a_k=\frac{2^k}{k!}$ and note that

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}}=\lim_{k\to\infty}\frac{k!2^{k+1}}{(k+1)!2^k}=\lim_{k\to\infty}\frac{2}{k+1}=0<1$$ and the series is convergent.

Example: Find all number $x>0$ for which the series

$$\sum_{k=1}^{\infty}k^3x^k=x+2^3x^2+3^3x^3+\cdots$$

converges.

Solution:

$$L=\lim_{k\to\infty}\frac{(k+1)^3x^{k+1}}{k^3x^k}=\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^3x=x$$

Thus, the series converges if $L=x<1$ and diverges if $x>1$. When $x=1$, the series becomes $\sum_{k=1}^{\infty}k^3$, which diverges by divergence test.

Root Test:

Given the series $\sum a_k$ with $a_k\geq 0$, suppose that $\lim_{k\to\infty}\sqrt[k]{a_k}=L$. The root test states the following:

If $L<1$, then $\sum a_k$ converges

If $L>1$ or L is infinite, then $\sum a_k$ diverges.

If $L=1$, the root test is conclusive.

Example: Test the series $\sum_{k=2}^{\infty}\frac{1}{(\ln k)^k}$ for convergence.

Solution:

Let $a_k=\frac{1}{(\ln k)^k}$ and note that

$$L=\lim_{k\to\infty}\sqrt[k]{a_k}=\lim_{k\to\infty}\sqrt[k]{(\ln k)^{-k}}=\lim_{k\to\infty}\frac{1}{\ln k}=0<1$$. Then, the series converges.

Example: Test the series $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k^2}$ for convergence.

Example: Test the series $\sum_{k=0}^{\infty}\frac{k!}{1\cdot 4\cdot 7\cdots(3k+1)}$ for convergence.

6. ALTERNATING SERIES; ABSOLUTE AND CONDITIONAL CONVERGENCE

There are two classes of series for which the successive terms alternate in sign, and each of these is appropriately called alternating series:

$$\sum_{k=1}^{\infty}(-1)^k a_k = -a_1 + a_2 - a_3 + \cdots$$

$$\sum_{k=1}^{\infty}(-1)^{k+1} = a_1 - a_2 + a_3 - a_4 + \cdots$$

where $a_k > 0$ in both cases.

Alternating Series Test

An alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ or $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ where $a_k > 0$, for all *k*, converges if both of the following two conditions are satisfied:

1/. $\lim_{k\to\infty} a_k = 0$

2/. $\{a_k\}$ is decreasing sequence; that is, $a_{k+1} \le a_k$ for all *k*.

Example: Determine if the following series is convergent or divergent. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$

Solution:

We have $\lim_{k\to\infty} b_k = \lim_{k\to\infty}\frac{1}{k} = 0$ and $b_k = \frac{1}{k} > \frac{1}{k+1} = b_{k+1}$. Hence the series is convergent.

Example: Investigate the series $\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{k^2+5}$

Example: Determine if the following series is convergent or divergent. $\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$

Absolutely And Conditionally Convergent Series

The series $\sum a_k$ is **absolutely convergent** if the related series $\sum |a_k|$ converges. The series $\sum a_k$ is **conditionally convergent** if it converges but $\sum |a_k|$ diverges.

The Generalized Ratio Test

For the series $\sum a_k$, suppose $a_k \ne 0$ for $k \ge 1$ and that

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L$$

where *L* is a real number or ∞, then

If $L < 1$, then the series $\sum a_k$ converges absolutely and hence converges.

If $L > 1$ or L infinite, the series $\sum a_k$ diverges.

If $L = 1$, the test is inconclusive.

Example: Determine if each of the following series are absolute convergent, conditionally convergent or divergent.

a. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ **b.** $\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n^2}$ **c.** $\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$

7. POWER SERIES

An infinite series of the form

$$\sum_{k=0}^{\infty}a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is called a **power series** in $(x-c)$. The number $a_0, a_1, a_2, \ldots$ are the *coefficients* of

$$\sum_{k=0}^{\infty}a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

which may be considered as an extension of a polynomial in *x*.

Convergence of a power series

For a power series $\sum_{k=0}^{\infty}a_k x^k$, exactly one of the following is true:

1. The series oversees for all *x*.
2. The series converges only for $x = 0$
3. The series **converges absolutely** for all *x* in an open interval $(-R, R)$ and **diverges** for $|x| > R$. It may either converge of diverge at the endpoints of the interval, $x = -R$ and $x = R$.

We call the interval $(-R, R)$ the **interval of convergence** of the power series. R is called **the radius of convergence** of the series. If the series converges only for $x = 0$, the series has radius of convergence $R = 0$ and if it converges for all x, we say that $R = \infty$.

Example: Show that the power series $\sum_{k=1}^{\infty}\frac{x^k}{k!}$ converges for all *x*.

Solution:

$$L = \lim_{k\to\infty}\left|\frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}}\right| = \lim_{k\to\infty}\left|\frac{x^{k++1}k!}{(k+1)!x^k}\right| = \lim_{k\to\infty}\frac{|x|}{k+1} = 0$$

Hence the series converges for all *x*.

Example: Determine the convergence set for the power series $\sum_{k=1}^{\infty}\frac{x^k}{\sqrt{k}}$

Solution:

By the generalized ratio test, we find

$$L = \lim_{k\to\infty}\left|\frac{\frac{x^{k+1}}{\sqrt{k+1}}}{\frac{x^k}{\sqrt{k}}}\right| = \lim_{k\to\infty}\left|\frac{\sqrt{k}}{\sqrt{k+1}}\right||x| = |x|$$

The power series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$.

For $x = -1$: $\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}$ converges by the alternation series test

For $x=1$: $\sum_{k=1}^{\infty}\frac{(1)^k}{\sqrt{k}}$ diverges

Thus, the given-above power series converges for $-1\le x<1$ and diverges otherwise.

Example: Find the interval of convergence for the power series $\sum_{k=1}^{\infty}\frac{2^k x^k}{k}$. What is the radius of convergence?

Example: Find the interval of convergence of the power series $\sum_{k=0}^{\infty}\frac{(x+1)^k}{3^k}$

Term-By-Term Differentiation and Integration Of Power Series

A power series $\sum_{k=0}^{\infty}a_k x^k$ with radius of convergence R can be differentiated or integrated term by term on its interval of absolute convergence $-R<x<R$. More specifically, if $\sum_{k=0}^{\infty}a_k x^k$ for $|x|<R$, then for $|x|<R$ we have

$$f'(x)=\sum_{k=1}^{\infty}ka_k x^{k-1}=a_1+2a_2x+3a_3x^2+\cdots$$

and

$$\int f(x)\,dx=\int\left(\sum_{k=0}^{\infty}a_k x^k\right)dx=\sum_{k=0}^{\infty}\left(\int a_k x^k\right)dx=\sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1}+C$$

Example: Let *f* be a function defined by the power series $f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}$ for all x

Show that $f'(x)=f(x)$ for all *x*, and deduce that $f(x)=e^x$

Solution:

$$\begin{aligned}f'(x)&=\frac{d}{dx}\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right]\\&=0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\cdots\\&=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\\&=f(x)\end{aligned}$$

If we have $f'(x)=f(x)$, then $f(x)=Ce^x$ and $f(0)=C$

$f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, then $f(0)=1+0+\frac{0^2}{2!}+\frac{0^3}{3!}+\cdots=1$

So we obtain $C=1$. Therefore $f(x)=e^x$.

8. **TAYLOR AND MACLAURIN SERIES**

Definition:

If f has derivatives of all orders at a, then we define the **Taylor series *f* about** $x = a$ **to be**

$$\sum_{0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots$$

Definition:

If f has derivatives of all orders at a, then we define the **Taylor series *f* about** $x = a$ **to be**

$$\sum_{0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

Example: Find the Maclaurin series for e^x, $\cos x$, and $\sin x$

Example: Find the Taylor series about $x = 1$ for $1/x$

9. TAYLOR’FOMULA WITH REMAINDER; CONVERGENCE OF TAYLOR SERIES

Taylor’s Theorem

Suppose that a function f can be differentiated $n+1$times at each point in an interval containing the point *a*, and let

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^k$$

be the nth Taylor polynomial about $x = a$ for f. Then for each *x* in the interval, there is at least one point *c* between *a* and *x* such that

$$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

We can rewrite $f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

then we can write $f(x)$ as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

and we call it **Taylor’s formula with remainder.**

Convergence of Taylor Series

The *Taylor series* for f converges to $f(x)$ at precisely those points where the remainder approaches zero; that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \Leftrightarrow \lim_{n\to+\infty} R_n(x) = 0$$

Constructing Maclaurin Series by Substitution

Sometimes Maclaurin series can be obtained by substituting in other Macluarin series

Example: Using the Maclaurin series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad -\infty < x < +\infty$

we can derive the Maclaurin series for e^{-x} by substituting $-x$ for x to obtain

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots \qquad -\infty < -x < +\infty$$

or $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \qquad -\infty < x < +\infty$

Example: Obtain the Maclaurin series for $1/(1-2x^2)$ by using the Maclaurin series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad -1 < x < 1$$