Lecture Note

Mathematics for Engineering I

Norton University
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Mathematics for Engineering I

Contents

Chapter 1

Matrix Algebra

1 Definition

A rectangular array of numbers is called a **matrix**. Each number is called the element or entry of the matrix. A *Row* of a matrix is a horizontal array and a **column** is a vertical one. The **size** or **order** of a matrix is determined by the number of rows and columns of that matrix. A matrix is said to be of **order** or **size** $m \times n$ if it has m rows and n columns. If $m = n$, the matrix is called the square matrix of order or size $n \times n$. Any real number is called a **scalar**.

There are a number of ways to represent an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[a_{ij} \right] = \left[a_{ij} \right]_{m \times n}$$

where a_{ij} with $i = 1, 2, ..., m; \ j = 1, 2, ..., n$ are called the elements or the entries of the matrix.

Example 1

$\begin{bmatrix} 2 & 4 & -9 \\ 6 & -8 & 1 \end{bmatrix}$ is a matrix of size 2×3. It has 2 rows and 3 columns.

$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ is a matrix of size 3×2 for it has 3 rows and 2 columns.

The matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a **square matrix** of size $n \times n$ or simply *n.*

The element a_{ij} where $i = j$ form a **main diagonal** of the matrix.

The matrix

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

is an $m \times 1$ matrix. Such a matrix is also called a **column vector** as for

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \end{bmatrix}$$

a $1 \times n$ matrix is called a **row vector**.

2 Some other matrices with special properties

Zero matrix or **null matrix** A matrix whose all elements equal zero.

Example 1

$$O_{4\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Upper triangular matrix A square matrix whose elements $a_{ij} = 0$ for all $i > j$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix A square matrix whose elements $a_{ij} = 0$ for all $i < j$.

$$\begin{bmatrix} a_{11} & & & \\ a_{12} & a_{22} & 0 & \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Diagonal matrix A square matrix whose elements $a_{ij} = 0$ for all $i \neq j$. It is both upper triangular and lower triangular matrix.

$$\begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{nn} \end{bmatrix}$$

Scalar matrix A diagonal matrix whose $a_{ij} = \lambda$ for $i = j$.

$$\begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ is a scalar matrix.

Identity matrix A scalar matrix whose $a_{ij} = 1$ for all $i = j$. An identity matrix of size $n \times n$ is commonly denoted by I_n.

Example 3

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric Matrix A matrix $A = \left[a_{ij}\right]_{n\times n}$ is symmetric to its main diagonal if $a_{ij} = a_{ji}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$.

Example 4

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 3 & 6 \\ 5 & 6 & 2 \end{bmatrix} \text{ is a symmetric matrix.}$$

3 Equality between two matrices

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$. $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j.

Example

Let $A = \begin{bmatrix} x & y \\ 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 5 \\ a & b \end{bmatrix}$. Given that $A = B$. Find a, b, x and y?

4 Arithmetic Operations

Addition Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, then $C = A + B = \left[c_{ij}\right]_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$ for all i and j.

Example 1

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$. Then

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 3+1 \\ 2+1 & -1+3 & 4+(-4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \end{bmatrix}$$

Properties of Matrix Addition

If the matrix A, B, C, and the null matrix O are of the same size, then

i. $A + B = B + A$ (Commutative Law)

ii. $A + (B + C) = (A + B) + C$ (Associative Law)

iii. $A + O = O + A = A$

iv. Each matrix A has a negative, $-A$, such that $A + (-A) = O$

Scalar Multiplication Let $A = \left[a_{ij}\right]_{m\times n}$ and r be a real number. Then the *scalar multiplication* of the matrix A and the scalar r

is a matrix C such that $C=\left[c_{ij}\right]_{m\times n}$ where $c_{ij}=ra_{ij}$ for all i and j.

Example 2

$$-2\begin{bmatrix}4 & -2 & -3\\ 7 & -3 & 2\end{bmatrix}=\begin{bmatrix}(-2)(4) & (-2)(-2) & (-2)(-3)\\ (-2)(7) & (-2)(-3) & (-2)(2)\end{bmatrix}=\begin{bmatrix}-8 & 4 & 6\\ -14 & 6 & -4\end{bmatrix}$$

Properties of Scalar Multiplication

Le r and s be real numbers and A, and B be matrices, then

i. $r(sA)=(rs)A$

ii. $(r+s)A=rA+sA$

iii. $r(A+B)=rA+rB$

iv. $A(rB)=r(AB)=(rA)B$

The proof (Exercises)

Example 3

Consider $A=\begin{bmatrix}4 & 2 & 3\\ 2 & -3 & 4\end{bmatrix}, B=\begin{bmatrix}3 & -2 & 1\\ 2 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}$, then

$$2(3A)=2\begin{bmatrix}12 & 6 & 9\\ 6 & -9 & 12\end{bmatrix}=\begin{bmatrix}24 & 12 & 18\\ 12 & -18 & 24\end{bmatrix}=6A$$

We also have

$$A(2B)=\begin{bmatrix}4 & 2 & 3\\ 2 & -3 & 4\end{bmatrix}\begin{bmatrix}6 & -4 & 2\\ 4 & 0 & -2\\ 0 & 2 & 4\end{bmatrix}=\begin{bmatrix}32 & -10 & 16\\ 0 & 0 & 26\end{bmatrix}=2(AB)$$

Dot Product The dot product $A\cdot B$ of a $1\times n$ row vector

$$A=\begin{bmatrix}a_1 & a_2 & \cdots & a_n\end{bmatrix}$$

and an $n\times 1$ column vector

$$B=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$$

is defined as

$$A\cdot B=\begin{bmatrix}a_1 & a_2 & \cdots & a_n\end{bmatrix}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}=a_1b_1+a_2b_2+\cdots+a_nb_n=\sum_{k=1}^{n}a_kb_k$$

Example 4

Dot product of $A=\begin{bmatrix}-2 & 1 & -5\end{bmatrix}$ and $B=\begin{bmatrix}-4\\ 3\\ -1\end{bmatrix}$ is defined by

$$A \cdot B = \begin{bmatrix} -2 & 1 & -5 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix}$$
$$= (-2)(-4) + (1)(3) + (-5)(-1)$$
$$= 8 + 3 + 5 = 16$$

Matrix Multiplication Let $A = \left[a_{ij} \right]_{m \times n}$ and $B = \left[b_{ij} \right]_{n \times p}$, then the product $AB = C = \left[c_{ij} \right]_{m \times p}$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$
$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

That is, the matrix product AB is a matrix whose element at (i, j) is the dot product of row i of the matrix A and column j of the matrix B.

Example 5

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$. Find AB.

Solution

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \\ 2 & 12 \end{bmatrix}$$

Properties of Matrix Multiplications

Let $A, B,$ and C be matrices, then

i. $A(BC) = (AB)C$

ii. $(A + B)C = AC + BC$

iii. $C(A + B) = CA + CB$

Example 6

Consider $A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 3 & 0 & -1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}$, then

$$BC = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 3 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 7 \\ 8 & -4 & 6 \\ 9 & 3 & 3 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 7 \\ 8 & -4 & 6 \\ 9 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}$$

$$AB=\begin{bmatrix}5 & 2 & 3\\ 2 & -3 & 4\end{bmatrix}\begin{bmatrix}2 & -1 & 1 & 0\\ 0 & 2 & 2 & 2\\ 3 & 0 & -1 & 3\end{bmatrix}=\begin{bmatrix}19 & -1 & 6 & 13\\ 16 & -8 & -8 & 6\end{bmatrix}$$

$$(AB)C=\begin{bmatrix}19 & -1 & 6 & 13\\ 16 & -8 & -8 & 6\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 2 & -3 & 0\\ 0 & 0 & 3\\ 2 & 1 & 0\end{bmatrix}=\begin{bmatrix}43 & 16 & 56\\ 12 & 30 & 8\end{bmatrix}$$

Hence,

$$A(BC)=(AB)C\,.$$

Example 7

If we have $A=\begin{bmatrix}2 & 2 & 3\\ 3 & -1 & 2\end{bmatrix}, B=\begin{bmatrix}0 & 0 & 1\\ 2 & 3 & -1\end{bmatrix}$, and $C=\begin{bmatrix}1 & 0\\ 2 & 2\\ 3 & -1\end{bmatrix}$, then

$$A+B=\begin{bmatrix}2 & 2 & 4\\ 5 & 2 & 1\end{bmatrix}$$

We obtain

$$(A+B)C=\begin{bmatrix}2 & 2 & 4\\ 5 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}18 & 0\\ 12 & 3\end{bmatrix}$$

$$AC=\begin{bmatrix}2 & 2 & 3\\ 3 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}15 & 1\\ 7 & -4\end{bmatrix}$$

$$BC=\begin{bmatrix}0 & 0 & 1\\ 2 & 3 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}3 & -1\\ 5 & 7\end{bmatrix}$$

$$AC+BC=\begin{bmatrix}15 & 1\\ 7 & -4\end{bmatrix}+\begin{bmatrix}3 & -1\\ 5 & 7\end{bmatrix}=\begin{bmatrix}18 & 0\\ 12 & 3\end{bmatrix},$$

Hence,

$$(A+B)C=AC+BC\,.$$

Transpose of a Matrix If $A=\left[a_{ij}\right]$ is an $m\times n$ matrix, then the transpose of A is $A^T=\left[a_{ji}\right]$ which is an $n\times m$ matrix. Thus the transpose of *A* is obtained from *A* by interchanging the rows and columns of *A*. We obtain

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}, \text{ then } A^T=\begin{bmatrix}a_{11} & a_{21} & \cdots & a_{m1}\\ a_{12} & a_{22} & \cdots & a_{m2}\\ \vdots & \vdots & & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{mn}\end{bmatrix}$$

Example 8

If $A=\begin{bmatrix}-3 & 4\\ 2 & 7\end{bmatrix}$, then $A^T=\begin{bmatrix}-3 & 2\\ 4 & 7\end{bmatrix}$

If $B=\begin{bmatrix}1 & 2 & -1\\ -3 & 2 & 7\end{bmatrix}$, then $B^T=\begin{bmatrix}1 & -3\\ 2 & 2\\ -1 & 7\end{bmatrix}$

$B^T=\begin{bmatrix}1 & -3\\ 2 & 2\\ -1 & 7\end{bmatrix}$, then $\left(B^T\right)^T=\begin{bmatrix}1 & 2 & -1\\ -3 & 2 & 7\end{bmatrix}=B$

Properties of transpose

Let s be a scalar and $A,$ and B be matrices. Then,

(i). $\left(A^T\right)^T=A$

(ii). $\left(A+B\right)^T=A^T+B^T$

(iii). $\left(AB\right)^T=B^TA^T$

(iv). $\left(sA\right)^T=sA^T$

Example 9

Let

$$A=\begin{bmatrix}1 & 2 & 3\\ -2 & 0 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}3 & -1 & 2\\ 3 & 2 & -1\end{bmatrix}, \text{ then}$$

$$A^T=\begin{bmatrix}1 & -2\\ 2 & 0\\ 3 & 1\end{bmatrix} \text{ and } B^T=\begin{bmatrix}3 & 3\\ -1 & 2\\ 2 & -1\end{bmatrix}$$

and also,

$$A+B=\begin{bmatrix}4 & 1 & 5\\ 1 & 2 & 0\end{bmatrix} \text{ and hence } \left(A+B\right)^T=\begin{bmatrix}4 & 1\\ 1 & 2\\ 5 & 0\end{bmatrix}$$

And we see that

$$A^T+B^T=\begin{bmatrix}1 & -2\\ 2 & 0\\ 3 & 1\end{bmatrix}+\begin{bmatrix}3 & 3\\ -1 & 2\\ 2 & -1\end{bmatrix}=\begin{bmatrix}4 & 1\\ 1 & 2\\ 5 & 0\end{bmatrix}=\left(A+B\right)^T$$

Example 10

Let $A=\begin{bmatrix}1 & 3 & 2\\ 2 & -1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ 2 & 2\\ 3 & -1\end{bmatrix}$, then

$$AB=\begin{bmatrix}1 & 3 & 2\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}0 & 1\\ 2 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}12 & 5\\ 7 & -3\end{bmatrix}$$

$$\left(AB\right)^T=\begin{bmatrix}12 & 5\\ 7 & -3\end{bmatrix}^T=\begin{bmatrix}12 & 7\\ 5 & -3\end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix} \text{ And } B^T = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = (AB)^T$$

Remark

a. If a and b are real numbers, then $ab = 0$ if and only if $a = 0$ or $b = 0$, but this is not true if A and B are matrices.

Example 11

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \text{ are not zero matrices, but}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b. If $a, b,$ and c are real numbers such that $ab = ac$ then $b = c$; that is, we can cancel out a leaving $b = c$, but this is not true in the case where $A, B,$ and C are matrices.

Example 12

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$$

we obtain

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix} \text{but } B \neq C.$$

c. Suppose that A is a square matrix. If p is a positive integer then we define $A^p = A \cdot A \cdots\cdot A$. If A is a square $n \times n$ matrix, then it is defined that $A^0 = I_n$.

d. Let p and q be nonnegative integers and A be a square matrix. Then $A^p A^q = A^{p+q}$ and $\left(A^p\right)^q = A^{pq}$.

e. If $A^{\mathrm{T}} = A$, then A is a **symmetric matrix**.

f. If $A^{\mathrm{T}} = -A$, then A is called a **skew symmetric matrix**.

Example 13

$$B = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} \text{is a skew symmetric matrix.}$$

5 Nonsingular Matrix

A square $n \times n$ matrix A is said to be **nonsingular** or **invertible** if there is another $n \times n$ matrix B such that

$$AB = BA = I_n$$

In this case the matrix B is called the **inverse** of A and is commonly denoted by A^{-1} rather than B. Otherwise, the matrix A is called **singular** or **noninvertible.**

Remark

i. To verify that B is an inverse of A, we need only verify that $AB = I_n$.

Example 1

Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$. Since $AB = BA = I_2$, we conclude that B is the inverse of A.

ii. The inverse of a matrix, if it exists, is unique.

Proof

Let B and C be inverses of A. Then $AB = BA = I_n$ and $AC = CA = I_n$

We then have $B = BI_n = B(AC) = (BA)C = I_nC = C$

Because of this uniqueness, we write the inverse of a nonsingular matrix A as A^{-1}. Thus

$$AA^{-1} = A^{-1}A = I_n$$

Example 2

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

If A^{-1} exists, let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we must have

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries of these two matrices, we obtain

$$\begin{aligned} a+2c &= 1 \\ 3a+4c &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} b+2d &= 0 \\ 3b+4d &= 1 \end{aligned}$$

Then $a = -2, c = \frac{3}{2}, b = 1,$ and $d = -\frac{1}{2}$

Moreover, since the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

also satisfies the properties that

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we conclude that A is a nonsingular and that

$$A^{-1}=\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$$

Example 3

Let

$$A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$$

If A^{-1} exists, let

$$A^{-1}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

Then we must have

$$AA^{-1}=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

so that

$$\begin{bmatrix}a+2c & b+2d\\ 2a+4c & 2b+4d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Equating corresponding entries of these two matrices, we obtain

$$\begin{cases}a+2c=1\\ 2a+4c=0\end{cases} \text{ and } \begin{cases}b+2d=0\\ 2b+4d=1\end{cases}$$

we cannot find $a,b,c,$ and d . So our assumption that A^{-1} exists is incorrect. Thus, A is singular.

Remark:

i. If A and B are both nonsingular $n\times n$ matrices, then AB is nonsingular and $(AB)^{-1}=B^{-1}A^{-1}$

Proof

We have $(AB)(B^{-1}A^{-1})=A(BB^{-1})A^{-1}=(AI_n)A^{-1}=AA^{-1}=I_n$. Similarly, $(B^{-1}A^{-1})(AB)=I_n$. Therefore, AB is nonsingular. Since the inverse of a matrix is unique, we conclude that $(AB)^{-1}=B^{-1}A^{-1}$

More generally, If $A_1,A_2,...,A_r$ are $n\times n$ nonsingular matrices, then $A_1A_2...A_r$ is nonsingular and $(A_1A_2...A_r)^{-1}=A_r^{-1}A_{r-1}^{-1}....A_1^{-1}$.

ii. If A is a nonsingular matrix, then A^{-1} is nonsingular and $(A^{-1})^{-1}=A$

iii. If A is a nonsingular matrix, then A^{T} is nonsingular and $(A^{-1})^{\mathrm{T}}=(A^{\mathrm{T}})^{-1}$.

Example 4

If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, then $A^{-1}=\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$ and $(A^{-1})^{T}=\begin{bmatrix}-2 & \frac{3}{2}\\ 1 & -\frac{1}{2}\end{bmatrix}$,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \text{ and } \left(A^T\right)^{-1} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \left(A^{-1}\right)^T$$

6 Echelon Form of a Matrix

Definition1

An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the following properties

i. All zero rows, if there are any, appear at the bottom of the matrix
ii. The first entry from the left of a nonzero row is a 1. This entry is called a **leading one** of its row.
iii. For each nonzero row, the leading one appears to the right and below any leading one's in preceding rows.
iv. If a column contains a leading one, then all other entries in that column are zero.

A matrix in reduced row echelon form appears as a staircase (echelon) pattern of leading 1s descending from the upper left corner of the matrix.
An $m \times n$ matrix satisfying properties i, ii, and iii is said to be in **row echelon form.**

Example 1

The following are matrices in reduced row echelon form

$$A = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition2:

An **elementary row (column) operation** on a matrix A is any one of the following operations:

a. **Type I:** Interchange any two rows (columns)
b. **Type II:** Multiply a row (column) by a nonzero number.
c. **Type III:** Add a multiple of one row (column) to another.

Example 2

Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$$

Interchange row 1 and row 3, we obtain

$$B = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Multiplying the 3rd of *A* by $\frac{1}{3}$, we obtain

$$C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

Adding (-2) times row 2 of *A* to row 3 of *A*, we obtain

$$D = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Definition3:
An $m \times n$ matrix *B* is said to be **row (column) equivalent** to an $m \times n$ *A* if *B* can be obtained by applying a finite sequence of elementary row (column) operations to *A*.

Example 3
Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{bmatrix}$$

If we add 2 times row 3 of *A* its second row, we obtain

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & -3 & 7 & 8 \\ 1 & -2 & 2 & 3 \end{bmatrix}$$

so *B* is row equivalent to *A*.
Interchanging rows 2 and 3 of *B*, we obtain

$$C = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & -1 & 2 & 3 \\ 4 & -3 & 7 & 8 \end{bmatrix}$$

so *C* is row equivalent to *B*.

7 Finding the Inverse of A matrix by Row Operations

If *A* is a nonsingular matrix, then by applying row operations we can transform matrix $\left[A \middle| I\right]$ to matrix $\left[I \middle| B\right]$. In this case *B* is the inverse of *A*. If the process cannot lead to matrix $\left[I \middle| B\right]$, we conclude the matrix *A* is singular. We summarize the process by the diagram below.

$$\begin{tikzcd} \left[A \middle| I\right] \arrow[r, "\text{Row Operations}"] & \left[I \middle| B\right] \arrow[d] \\ & \text{The inverse of } A \end{tikzcd}$$

Example 4

Find the inverse of $A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$

Solution

We write the matrix $\left[A\middle|I\right]$ then applying row operations as follows

$$\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 2 & 4 & 0 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 0 & -2 & -2 & 1\end{array}\right] R_2^{new}=(-2)R_1^{old}+R_2^{old}$$

$$\sim\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 0 & 1 & 1 & -\frac{1}{2}\end{array}\right] R_2^{new}=-\frac{1}{2}R_2^{old}$$

$$\sim\left[\begin{array}{cc|cc}1 & 0 & -2 & \frac{3}{2}\\ 0 & 1 & 1 & -\frac{1}{2}\end{array}\right] \quad R_1^{new}=(-3)R_2^{old}+R_1^{old}$$

Check

$$\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\begin{bmatrix}-2 & \frac{3}{2}\\ 1 & -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2$$

$$\begin{bmatrix}-2 & \frac{3}{2}\\ 1 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_2$$

Hence $A^{-1}=\begin{bmatrix}-2 & \frac{3}{2}\\ 1 & -\frac{1}{2}\end{bmatrix}$

Example 5

Find the inverse of $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 1 & 2 & 4\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 1 & 2 & 4 & 0 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right] R_3^{new}=(-1)R_1^{old}+R_3^{old}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & 0 & 4 & 0 & -3\\ 0 & 2 & 0 & 3 & 1 & -3\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right] \begin{array}{l}R_1^{new}=(-3)R_3^{old}+R_1^{old}\\ R_2^{new}=(-3)R_3^{old}+R_2^{old}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & 0 & 4 & 0 & -3\\ 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{3}{2}\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right] R_2^{new}=\left(\frac{1}{2}\right)R_2^{old}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad R_1^{new} = (-2)R_2^{old} + R_1^{old}$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ \dfrac{3}{2} & \dfrac{1}{2} & -\dfrac{3}{2} \\ -1 & 0 & 1 \end{bmatrix}$$

Example 6

Find the inverse of $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right]$$

Hence A is singular.

Exercises

1. Let

$$A_1 = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

a. Calculate $2A_1 - 3A_2 + 4A_3$

b. Write $c_1A_1 + c_2A_2 + c_3A_3$ as a single matrix

2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$$

a. Which pairs of matrices can be added?

b. Calculate $3A + 4B$

c. Write a matrix F such that $D + F = D$

d. Write a matrix G such that $D + G = \mathbf{0}_{2\times 2}$

e. Find a matrix H such that $D + H = E$

3. Let

$$A=\begin{bmatrix}1 & 3 & 1\\ 3 & 1 & 2\end{bmatrix} \text{ and } B=\begin{bmatrix}-1 & 1 & 0\\ 1 & 4 & 3\end{bmatrix}$$

a. Calculate $2A+3B$
b. Calculate $A-B$
c. Find C so that $A-B+C=\mathbf{0}$
d. What is the size of the matrix $\mathbf{0}$ in c.?

4. Let

$$A=\begin{bmatrix}1 & -4\\ 2 & 5\\ 3 & 7\end{bmatrix} \text{ and } B=\begin{bmatrix}0 & 1\\ 1 & 5\\ 2 & -1\end{bmatrix}$$

a. Calculate $A+B$
b. Write A^T and B^T
c. Calculate A^T+B^T
d. Write $(A+B)^T$
e. Compare $(A+B)^T$ and A^T+B^T

5. Let

$$A=\begin{bmatrix}-1 & 4\\ 2 & -1\end{bmatrix} \text{ and } B=\begin{bmatrix}3 & 2\\ 1 & 0\end{bmatrix}$$

Verify that $(A+B)^T=A^T+B^T$

6. Find *a*, *b* and *c* such that

$$\begin{bmatrix}1 & 2 & -1 & 3\\ 1 & 4 & 1 & 5\end{bmatrix}\begin{bmatrix}a & 3\\ 1 & 1\\ 0 & 2\\ 1 & 2\end{bmatrix}=\begin{bmatrix}9 & c\\ b & 19\end{bmatrix}$$

7. Calculate each of the following when it is defined, given that

$$A=\begin{bmatrix}1 & 2 & 1\\ 2 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}, B=\begin{bmatrix}-1 & 1 & 0\\ 0 & 1 & 1\\ 1 & -1 & -1\end{bmatrix}, C=\begin{bmatrix}1 & 1\\ 2 & 1\\ 0 & 4\end{bmatrix} \text{ and } D=\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

a. *AB* **b.** *AC* **c.** *CD* **d.** *BC* **e.** *DC* **f.** *BC*+*CD* **g.** (*A*+*B*)*C* **h.** *AC*+*BC*

8. Let

$$A=\begin{bmatrix}1 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}1 & -1 & 2\\ 1 & 0 & 3\\ 1 & 1 & 0\end{bmatrix}$$

Calculate AB and B^TA^T. How are these matrices related? Can you calculate A^TB^T?

9. **a.** Calculate the product

$$\begin{bmatrix}3 & 5\\ -1 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

b. Write as a product of two matrices the 2×1 array $\begin{bmatrix}3x+4y\\ 2x-y\end{bmatrix}$

10. Let

$$A=\begin{bmatrix}2&1&1\\1&0&1\\0&1&-1\end{bmatrix}, B=\begin{bmatrix}2&1&3&7\\1&0&8&2\\1&4&3&6\end{bmatrix}, C=\begin{bmatrix}3&-1&8&4\\0&2&3&5\\0&6&-2&9\end{bmatrix}$$

Calculate AB and AC. If A, B and C are matrices such that $AB=AC$ is it necessarily true that $B=C$? Justify your answer.

11. Let

$$D=\begin{bmatrix}2&0&0\\0&4&0\\0&0&-1\end{bmatrix},\ E=\begin{bmatrix}3&0&0\\0&-1&0\\0&0&2\end{bmatrix}$$

a. Calculate $D+E$

b. Calculate $aD+bE$ where a and b are scalars.

c. Find a matrix B such that $D+B=\mathbf{0}_{3\times3}$

12. Let

$$D=\begin{bmatrix}2&0&0\\0&4&0\\0&0&-1\end{bmatrix} \text{ and } A=\begin{bmatrix}1&2&1\\0&3&4\\2&1&1\end{bmatrix}$$

Calculate DA and AD

13. Compute

$$\begin{bmatrix}1&0&0\\-1&3&0\\3&2&-1\end{bmatrix}\begin{bmatrix}4&0&0\\3&1&0\\1&2&1\end{bmatrix}$$

14. If A and B are 3×3 lower triangular matrices, prove that AB is also 3×3 lower triangular matrix.

15. If $A=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$. Find A^2 and A^3. Can you suggest a formula for A^n?

16. **a.** Let $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}$. Calculate A^2, A^3, A^4. What is A^n for any integer $n\geq1$?

b. Let $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$. Calculate A^2, A^3, A^4. What is A^n?

c. Let $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$. What is A^n?

17. Find the inverse of $A=\begin{bmatrix}1&3\\2&4\end{bmatrix}, B=\begin{bmatrix}1&2&3\\0&2&3\\1&2&4\end{bmatrix}$

18. Which of the following matrices are singular? For the nonsingular ones find the inverse

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

19. Invert the following matrices, if possible

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 3 & -3 & -2 \\ 2 & 0 & 1 & 5 \\ 3 & 1 & -2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

20. Find the inverse, if it exists, of:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 1 & 2 & 1 & -2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

21. If A is a nonsingular matrix whose inverse is $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$ Find A.

22. If $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, find A.

Chapter 2

Determinant

1 Definition

Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then the **determinant** of the matrix A which is denoted by $\det(A)$ or $|A|$ is defined by $\det(A)=a_{11}a_{22}-a_{21}a_{12}$.

Example 1

Find the determinant of $A=\begin{bmatrix} 5 & 6 \\ -1 & 2 \end{bmatrix}$

Solution

$$\det(A)=\begin{vmatrix} 5 & 6 \\ -1 & 2 \end{vmatrix}=(5)(2)-(-1)(6)=16$$

The determinant of $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is defined by

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{32}a_{21}$$
$$-a_{31}a_{22}a_{13}-a_{21}a_{12}a_{33}-a_{11}a_{23}a_{32}$$

Example 2

Find the determinant of $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}=(2)(2)(2)+(1)(1)(0)+(3)(1)(3)-(0)(2)(3)-(3)(1)(2)-(2)(1)(1)$$
$$=8+0+9-0-6-2=9$$

2 Cofactor Expansions

Definition 1

Let $A=\left[a_{ij}\right]$ be an $n\times n$ matrix. Let M_{ij} be the $(n-1)\times(n-1)$ sub-matrix of A obtained by deleting the ith row and jth column of A. The determinant $\det\left(M_{ij}\right)$ is called the **minor** of a_{ij}.

Definition2

Let $A=\left[a_{ij}\right]$ be an $n\times n$ matrix. The **cofactor** A_{ij} of a_{ij} is defined as $A_{ij}=(-1)^{i+j}\det\left(M_{ij}\right)$.

Example 1

$$\text{Let } A=\begin{bmatrix}3 & -1 & 2\\ 4 & 5 & 6\\ 7 & 1 & 2\end{bmatrix}$$

Then

$$\det\left(M_{12}\right)=\begin{vmatrix}4 & 6\\ 7 & 2\end{vmatrix}=8-32=-34 \qquad \det\left(M_{23}\right)=\begin{vmatrix}3 & -1\\ 7 & 1\end{vmatrix}=3+7=10$$

and

$$\det\left(M_{31}\right)=\begin{vmatrix}-1 & 2\\ 5 & 6\end{vmatrix}=-6-10=-16$$

Also,

$$A_{12}=(-1)^{1+2}\det\left(M_{12}\right)=(-1)(-34)=34$$

$$A_{23}=(-1)^{2+3}\det\left(M_{23}\right)=(-1)(10)=-10$$

and

$$A_{31}=(-1)^{3+1}\det\left(M_{31}\right)=(1)(-16)=-16$$

If we think of the sign $(-1)^{i+j}$ as being located in position (i,j) of an $n\times n$ matrix, then the signs form a checkerboard pattern that has a $+$ in $(1,1)$ position.

The patterns for $n=3$ and $n=4$ are as follows:

$$\underset{n=3}{\begin{bmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{bmatrix}} \qquad \underset{n=4}{\begin{bmatrix}+ & - & + & -\\ - & + & - & +\\ + & - & + & -\\ - & + & - & +\end{bmatrix}}$$

Definition3

Let $A=\left[a_{ij}\right]$ be an $n\times n$ matrix. Then the *expansion of* $\det(A)$ *along the* ith row is defined by

$$\det(A)=|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}$$

and the *expansion of* $\det(A)$ *along the* jth *column* is defined by

$$\det(A)=|A|=a_{1j}A_{1j}+a_{2j}A_{2j}+\cdots+a_{nj}A_{nj}$$

Example 2

Evaluate the determinant $\begin{vmatrix} 1 & 2 & -2 \\ 3 & 0 & -1 \\ 2 & 1 & 4 \end{vmatrix}$

By the expansion along row 2, we obtain

$$\begin{vmatrix} 1 & 2 & -2 \\ 3 & 0 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 3\cdot A_{21} + 0\cdot A_{22} + (-1)\cdot A_{23}$$

$$= 3\times(-1)^{2+1}\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} + (-1)\times(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (-3)\times 10 - 3$$

$$= -33$$

Example 3

Evaluate the determinant $\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$ (Ans: 48).

3 Adjoint Matrix

Definition

Let $A = [a_{ij}]$ be an $n\times n$ matrix. The $n\times n$ matrix $adjA$ called the *adjoint* of A is the matrix whose (i, j) th entry is the cofactor A_{ji} of a_{ji}. Thus

$$adjA = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Example 1

Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$. Find the adjoint of A.

Solution

We first compute the cofactors of A. We have

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18 \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17 \quad A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6 \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10 \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28$$

Then

$$adjA = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

Remark

i. If $A = \left[a_{ij}\right]$ is an $n \times n$ matrix, then $A(adjA) = (adjA)A = \det(A)I_n$.

Example 2

Let be a matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$. Then the determinant of A is

$$\det(A) = 4\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 12 + 2 = 14$$

And the cofactors of A are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \;\; A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2 \;\; A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = -6 \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2 \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} = -8 \quad A_{33} = (-1)^{3+3}\begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix} = 4$$

Then

$$\mathrm{adj}A = \begin{bmatrix} 3 & -6 & 2 \\ 2 & 10 & -8 \\ -1 & 2 & 4 \end{bmatrix}$$

Therefore

$$A(adjA) = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & -6 & 2 \\ 2 & 10 & -8 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 2 \\ 2 & 10 & -8 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$= 14\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det(A)I_3$$

ii. If A is a square matrix whose determinant is different from zero, the inverse of A can be found by the formula

$$A^{-1} = \frac{1}{\det(A)}(\text{adj}A)$$

Example 3

The inverse of $A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ is

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & -6 & 2 \\ 2 & 10 & -8 \\ -1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{14} & -\frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{5}{7} & -\frac{4}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

iii. The value of a determinant remains unchanged if we add a multiple of a row (column) to another.

Example 4

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & -3 & 5 & 1 \\ -2 & 4 & 1 & -4 \\ 3 & 4 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 7 & -5 \\ 0 & 8 & -1 & 2 \\ 0 & -2 & 1 & -1 \end{vmatrix} = (1)\begin{vmatrix} -7 & 7 & -5 \\ 8 & -1 & 2 \\ -2 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -7 & 7 & -5 \\ 8 & -1 & 2 \\ -2 & 1 & -1 \end{vmatrix} = (-7)\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - 7\begin{vmatrix} 8 & 2 \\ -2 & -1 \end{vmatrix} - 5\begin{vmatrix} 8 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= -7(-1) - 7(-4) - 5(6) = 5$$

Example 5

Evaluate $\begin{vmatrix} 4 & 1 & 3 & 6 & 2 \\ 0 & -1 & 1 & 3 & 2 \\ 8 & 3 & 4 & 9 & 0 \\ 8 & 2 & 4 & 6 & 4 \\ 3 & 0 & -1 & 5 & 2 \end{vmatrix}$ (Answer: -140)

4 Some More Remarks on Determinants

i. If A is a square matrix then $\det(A) = \det(A^T)$.

Example 1

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

$\det(A) = -13$ and $\det(A^T) = -13$

ii. A is a square matrix. If we multiply a row or a column of a matrix by a real number u, the determinant of the matrix obtained equals the product of u and determinant of A.

Example 2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} ua & ub \\ c & d \end{vmatrix} = uad - ubc = u\left(ad - bc\right)$$

$$\begin{vmatrix} ua & b \\ uc & d \end{vmatrix} = uad - ubc = u\left(ad - bc\right)$$

iii. If A is a square matrix with two identical rows of columns, then $\det\left(A\right) = 0$.

iv. If A is a square matrix with a zero row or zero column, then $\det A = 0$.

v. If A is a triangular matrix (upper triangular or lower triangular) then the determinant of A is the product of the main diagonal elements.

vi. If A^{-1} exists, then $\det\left(A^{-1}\right) = \dfrac{1}{\det A}$

vii. If A and B are $n \times n$, $\det\left(AB\right) = \left(\det A\right)\left(\det B\right)$

Example 3

Let

$$A = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \text{and } B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

then

$$\det A = \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 6 \text{ and } \det B = \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

and

$$\left(\det A\right)\left(\det B\right) = 6 \times \left(-2\right) = -12$$

$$AB = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -1 & -2 \end{bmatrix}$$

$$\det\left(AB\right) = \begin{vmatrix} 7 & 2 \\ -1 & -2 \end{vmatrix} = -12$$

Exercises

1. Evaluate the determinants

a. $\begin{vmatrix} -2 & 7 \\ -3 & 5 \end{vmatrix}$ (Ans: 11) b. $\begin{vmatrix} 15 & -2 \\ -5 & 2 \end{vmatrix}$ (Ans: 20) c. $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \\ -5 & 7 & 2 \end{vmatrix}$ (Ans: 18)

d. $\begin{vmatrix} 3 & -5 & 7 \\ 1 & 9 & 0 \\ -2 & 1 & 3 \end{vmatrix}$ (Ans: 229) e. $\begin{vmatrix} 7 & 9 & 15 \\ 4 & 8 & 3 \\ 2 & 4 & 0 \end{vmatrix}$ (Ans:-30) f. $\begin{vmatrix} 2 & -1 & 3 \\ 7 & 2 & -3 \\ 1 & 4 & 6 \end{vmatrix}$ (Ans: 171)

g. $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 5 \\ -1 & 3 & 1 \end{vmatrix}$ (Ans: 22) h. $\begin{vmatrix} 3 & 2 & 3 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{vmatrix}$ (Ans: -2) i. $\begin{vmatrix} -1 & 2 & 1 \\ 4 & 0 & 7 \\ 1 & 0 & -2 \end{vmatrix}$ (Ans: 30)

j. $\begin{vmatrix} 4 & 5 & 3 & 1 \\ 2 & 0 & 1 & 5 \\ -4 & 2 & 6 & 3 \\ -3 & 1 & -4 & 0 \end{vmatrix}$ (Ans: -1155)

2. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$.

a. Find $adjA$ b. Compute $\det A$ c. $A(adjA) = (adjA)A = \det(A)I_3$

3. Like problem **2.** for the matrix $A = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix}$

4. Use the formula $A^{-1} = \dfrac{1}{\det(A)}(adjA)$ to find the inverse of each of the following matrices if exists.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

5. The characteristic equation of a square matrix A is the equation $|A - \lambda I| = 0$.

Given the matrix $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, the characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 14$$

It can be shown that a square matrix always satisfies its characteristic equation. So, in this case $A^2 - 7A + 14I = \mathbf{0}$. Find the characteristic equation and demonstrate that the matrix satisfies the equation

$$A = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 1 & 5 \\ -1 & 5 & 2 \end{bmatrix}$$

6. The equation of line passing through the points (x_1, y_1) and (x_2, y_2) can be determined by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Find the equation of a line passing through

a. $(0,0)$ and $(5,3)$ **b.** $(0,0)$ and $(-2,2)$ **c.** $(-1,0)$ and $(5,-3)$

d. $(4,1)$ and $(-2,2)$ **e.** $(-4,3)$ and $(2,1)$ **f.** $(0,7)$ and $(2,-7)$

g. $\left(-\frac{1}{2},-3\right)$ and $\left(-\frac{5}{2},1\right)$

7. The area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Find the area of the triangle with the given vertices

a. $(0,0),(3,1),(1,5)$ **b.** $(-2,-3),(2,-3),(0,4)$

c. $(-1,2),(-3,1),(1,-5)$ **d.** $(-1,-2),(-3,1),(4,-5)$

e. $(1,-2),(-3,2),(4,-3)$ **f.** $(1,1),(-3,3),(4,-3)$

g. $(5,-1),(-3,3),(4,-3)$

8. Show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

Chapter 3

System of Linear Equations

1 Introduction

What is a linear equation?

An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (\text{ I })$$

which expresses b in terms of the unknowns $x_1, x_2, ..., x_n$ and the constants $a_1, a_2, ..., a_n$ is called **a linear equation**.

A **solution** to linear equation (I) is an array of n numbers $s_1, s_2, \dots s_n$ which has the property that (I) is satisfied when $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ are substituted in (I). For example $x_1 = 2, x_2 = 3,$ and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13$$

because

$$6(2) - 3(3) + 4(-4) = -13$$

More generally, a **system of *m* linear equations in *n* unknowns** $x_1, x_2, \dots, x_n$ or a **linear system** is a set of m linear equations, each in n unknowns. A linear system can conveniently be written as

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = b_1 \\ a_{21}x_2 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = b_m \end{cases} \qquad (\text{ II })$$

or in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$, called the **coefficient matrix**,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ vector of unknowns}$$

and

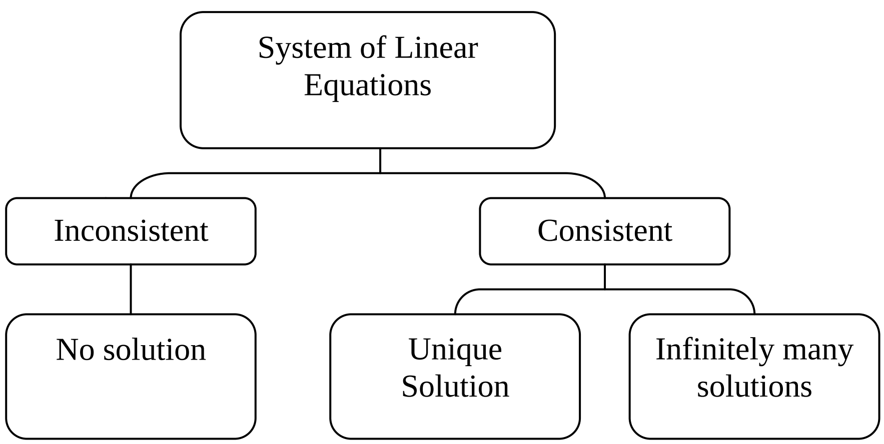
$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \text{ right hand side}$$

then the system can be written as

$$AX = B$$

The matrix $\left[A | B \right]$ or $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ is called the **augmented matrix**.

A system of linear equation which has solution is said to be **consistent**. Otherwise, is said to be **inconsistent**.



If $b_1 = b_2 = \cdots = b_m = 0$ then (II) becomes

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = 0 \\ a_{21}x_2 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = 0 \end{cases}$$

It is called a linear **homogeneous system**.

Note that $x_1 = x_2 = \cdots = x_n = 0$ is always a solution to a homogeneous system*;* it is called the **trivial solution**. A solution to a homogeneous system in which not all of $x_1, x_2, \ldots, x_n$ are zero is called a **nontrivial solution**.

2 Finding a Solution to a Linear System

We will discuss three methods for finding the solution to a system.

2.1 Gaussian Elimination and Gauss-Jordan Reduction

Given the system $AX = B$, to solve this system by Gaussian elimination method, we transform the augmented matrix $\left[A|B\right]$ to the matrix $\left[C|D\right]$ which is a row echelon matrix by using elementary row operations. Then to find the solution of the system from the corresponding augmented matrix $\left[C|D\right]$ we **back substitute.**

Example 1

Solve the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 - x_3 = 3 \end{cases}$$

Solution

The corresponding matrix $\left[A|B\right]$ of the system is

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

By elementary row operations, we transform this matrix to a row echelon matrix $\left[C|D\right]$.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 2 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \begin{array}{l} \\ R_2^{new} = (-2)R_1^{old} + R_2^{old} \\ R_3^{new} = (-3)R_1^{old} + R_3^{old} \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right] R_2^{new} = -\frac{1}{5} R_2^{old}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \quad R_3^{new} = 6R_2^{old} + R_3^{old}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_3^{new} = -\frac{1}{4} R_3^{old}$$

Using back substitution, we now obtain

$$x_3 = 3$$
$$x_2 = 2 - x_3 = -1$$
$$x_1 = 9 - 2x_2 - 3x_3 = 2$$

Hence, the solution is $x_1 = 2, x_2 = -1, x_3 = 3$

Example 2

Solve the system

$$\begin{cases} 2x_1 + x_2 - x_3 = -2 \\ x_1 - 2x_2 + x_3 = 2 \\ 4x_1 + 2x_2 - 2x_3 = 1 \end{cases}$$

Solution

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -2 \\ 1 & -2 & 1 & 2 \\ 4 & 2 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 2 & 1 & -1 & -2 \\ 4 & 2 & -2 & 1 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 5 & -3 & -6 \\ 0 & 10 & -6 & 7 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 5 & -3 & -6 \\ 0 & 0 & 0 & 19 \end{array}\right]$$

We conclude that the system has no solution since the last equation is

$$0x_1 + 0x_2 + 0x_3 = 19$$

which can never be satisfied.

Example 3

Solve the system

$$\begin{cases} x_1 + 2x_2 - 3x_3 = -4 \\ 2x_1 + x_2 - 3x_3 = 4 \end{cases}$$

The system is equivalent to

$$\begin{cases} x_1 + 2x_2 - 3x_3 = -4 \\ \quad -3x_2 + 3x_3 = 12 \end{cases}$$

or

$$\begin{cases} x_1 + 2x_2 - 3x_3 = -4 \\ \quad x_2 - x_3 = -4 \end{cases}$$

then

$$\begin{aligned} x_2 &= x_3 - 4 \\ x_1 &= -4 - 2x_2 + 3x_3 \\ &= -4 - 2\left(x_3 - 4\right) + 3x_3 \\ &= x_3 + 4 \end{aligned}$$

where x_3 can take on any real numbers. Thus a solution is of the form

$x_1 = r + 4, x_2 = r - 4, x_3 = r$ where r is any real number.

Example 4

Consider the linear system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases}$$

We form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

then by row operations we obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From the last augmented matrix, we obtain $x_1 = 1, x_2 = -2, x_3 = 3$, the solution of the system. This solution is obtained by applying **Gaussian Elimination Method**.

To solve the linear system by **Gauss-Jordan reduction**, we transform the last matrix to $\left[C \middle| D\right]$, which is in reduced row echelon form, by the following steps:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus the solution is $x_1 = 1, x_2 = -2, x_3 = 3$.

System $AX = B$,

$$\left[A \middle| B\right] \xrightarrow{\text{Row operations}} \left[C \middle| D\right]$$

Augmented matrix of the system.

row echelon matrix

Gaussian Elimination Method

$$\left[A \middle| B\right] \xrightarrow{\text{Row operations}} \left[C \middle| D\right]$$

Augmented matrix of the system.

reduced row echelon matrix

Gauss-Jordan Elimination Method

Example5 Solve the system

$$\begin{cases} 2x + 7y + 15z = -12 \\ 4x + 7y + 13z = -10 \\ 3x + 6y + 12z = -9 \end{cases} \qquad (x = 1, y = -2 \text{ and } z = 0)$$

Remark

In both **Gaussian elimination** and **Gauss-Jordan** reduction, we use row operations only. Do not try to use any column operations.

Homogeneous Systems

Now we consider a homogeneous system $AX = \mathbf{0}$ of m linear equations in n unknowns.

Example 6

Consider the homogeneous system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since the augmented matrix is in reduced row echelon form, the solution is seen to be

$$x_1 = -2r, x_2 = s, x_3 = -3r, x_4 = -4r, \text{and } x_5 = r$$

where r and s are any real numbers.

Remark

i. A homogeneous system of m linear equations in n unknowns always has a nontrivial solution if $m < n$, that is, if the number of unknowns exceeds the number of equations.
ii. If A is $m \times n$ and $AX = \mathbf{0}$ has only the trivial solution, then $m \geq n$.

Example7

Consider the homogeneous system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_4 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases}$$

The augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Hence the solution is $x_1 = -r, x_2 = r, x_3 = -r, \text{and } x_4 = r$, r any real number.

2.2 Solving the system by the inverse

If A is an $n \times n$ matrix, then the linear system $AX = B$ is a system of n equations in n unknowns. Suppose that A is a nonsingular. Then, A^{-1} exists and we can multiply

$$AX = B$$

by A^{-1} on both sides, we obtain

$$A^{-1}\left(AX\right) = A^{-1}B$$

or

$$I_n X = X = A^{-1}B$$

Moreover, $X = A^{-1}B$ is clearly a solution to the given linear system. Thus, if A is nonsingular, we have a unique solution.

Example 8

Consider the system $\begin{cases} 2x - 3y + 4z = 9 \\ x - 2y - z = 8 \\ y + 5z = 7 \end{cases}$

The matrix equation corresponding to the system is

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & -2 & -1 \\ 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -2 & -1 \\ 0 & 1 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$

Since $A^{-1} = \begin{bmatrix} -9 & 19 & 11 \\ -5 & 10 & 6 \\ 1 & -2 & -1 \end{bmatrix}$, the solution to the system is

$$X = A^{-1}B = \begin{bmatrix} -9 & 19 & 11 \\ -5 & 10 & 6 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 22 \\ -14 \\ 43 \end{bmatrix}$$

Example 9

Solve the system $\begin{cases} x_1 - x_2 + x_3 = k_1 \\ 2x_2 - x_3 = k_2 \\ 2x_1 + 3x_2 = k_3 \end{cases}$

For **a.** $k_1 = 1, k_2 = 1, k_3 = 1$ **b.** $x_1 = 3, k_2 = 1, k_3 = 4$ **c.** $k_1 = -5, k_2 = 2, k_3 = -3$.

2.3 Cramer's Rule

Suppose we have

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

a linear system of *n* equations in *n* unknowns. Let $A = \left[a_{ij}\right]$ be a coefficient matrix.

If $\det A \neq 0$ then the system has a unique solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \cdots, x_n = \frac{\det(A_n)}{\det(A)}$$

where A_i is a matrix obtained from *A* by replacing column *i* of *A* with right hand side vector.

Example 10

Consider the system $\begin{cases} 2x_1 - 3x_2 = 7 \\ -3x_1 + x_2 = -7 \end{cases}$

then

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix},\ \det(A) = -7 \quad A_1 = \begin{bmatrix} 7 & -3 \\ -7 & 1 \end{bmatrix}, \det(A_1) = -14$$

$$A_2 = \begin{bmatrix} 2 & 7 \\ -3 & -7 \end{bmatrix}, \det(A_2) = 7$$

Hence $x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{-14}{-7} = 2$ and $x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{7}{-7} = -1$.

Example 11

Solve the system $\begin{cases} 3x - z = 5 \\ x - y + z = 0 \\ x + y = 0 \end{cases}$

Exercises

1. Solve each system of the equations using matrices (row operations). If the system has no solution, say that it is inconsistent.

a. $\begin{cases} x - y = 6 \\ 2x - 3z = 16 \\ 2y + z = 4 \end{cases}$ b. $\begin{cases} x - 2y + 3z = 7 \\ 2x + y + z = 4 \\ -3x + 2y - 2z = -10 \end{cases}$ c. $\begin{cases} 2x + y - 3z = 0 \\ -2x + 2y + z = -7 \\ 3x - 4y - 3z = 7 \end{cases}$

d. $\begin{cases} 2x - 2y - 2z = 2 \\ 2x + 3y + z = 2 \\ 3x + 2y = 0 \end{cases}$ e. $\begin{cases} 2x - 3y - z = 0 \\ -x + 2y + z = 5 \\ 3x - 4y - z = 1 \end{cases}$ f. $\begin{cases} -x + y + z = -1 \\ -x + 2y - 3z = -4 \\ 3x - 2y - 7z = 0 \end{cases}$

g. $\begin{cases} 2x - 3y - z = 0 \\ 3x + 2y + 2z = 2 \\ x + 5y + 3z = 2 \end{cases}$ h. $\begin{cases} 2x - 2y + 3z = 6 \\ 4x - 3y + 2z = 0 \\ -2x + 3y - 7z = 1 \end{cases}$ i. $\begin{cases} 3x - 2y + 2z = 6 \\ 7x - 3y + 2z = -1 \\ 2x - 3y + 4z = 0 \end{cases}$

j. $\begin{cases} 3x+y-z=\dfrac{2}{3} \\ 2x-y+z=1 \\ 4x+2y=\dfrac{8}{3} \end{cases}$ k. $\begin{cases} x+2y+z=1 \\ 2x-y+2x=2 \\ 3x+y+3x=3 \end{cases}$ l. $\begin{cases} x-y+z=5 \\ 3x+2y-2z=0 \end{cases}$

m. $\begin{cases} 2x+y-z=4 \\ -x+y+3z=1 \end{cases}$ n. $\begin{cases} 2x+y-z=6 \\ x-y-z=1 \end{cases}$ o. $\begin{cases} x-y+2z=3 \\ 2x+y-2z=3 \end{cases}$

p. $\begin{cases} x-y+z=0 \\ 3x+y-5z=0 \\ -x-y+3z=0 \end{cases}$ q. $\begin{cases} x+2y-z=0 \\ 3x+7y+3z=0 \\ -x+4y+2z=0 \end{cases}$ r. $\begin{cases} x-y+3z=5 \\ 4y-4z=0 \\ -x+2y+3z=2 \end{cases}$

2. Find the parabola $y=ax^2+bx+c$ that passes through the points $(1,2),(-2,-7),(2,-3)$.

3. Find the function $f(x)=ax^3+bx^2+cx+d$ for which $f(-3)=-112,\ f(-1)=-2,\ f(1)=4,\ f(2)=13$.

4. Find the solution of the system using the inverse of matrix

a. $\begin{cases} 2x+3y=12 \\ x+2y=7 \end{cases}$ b. $\begin{cases} 5x+3y=16 \\ 2x+y=6 \end{cases}$ c. $\begin{cases} 4x+3y=2 \\ 2x+2y=4 \end{cases}$ d. $\begin{cases} x+13y=4 \\ x+11y=2 \end{cases}$

e. $\begin{cases} 2x+3z=2 \\ 3x-2y=-8 \\ 2x+y+3z=3 \end{cases}$ f. $\begin{cases} x+2y-2z=-1 \\ 3x+y+4z=17 \\ 5x-3y+z=2 \end{cases}$ g. $\begin{cases} 2x+3z=2 \\ -3x+y+2z=-14 \\ 4x-3y-7z=24 \end{cases}$

h. $\begin{cases} x-3y+2z=4 \\ 2x-7y+3z=5 \\ 3x-2y-7z=-7 \end{cases}$ i. $\begin{cases} x+y+z=4 \\ 2x-y-z=2 \\ x+2y+3z=3 \end{cases}$ j. $\begin{cases} 3x+y-2z=1 \\ 5x-y+4z=7 \\ x-y+z=3 \end{cases}$

k. $\begin{cases} x+3y+2z=9 \\ x+2y+2z=7 \\ 2x+6y+3z=17 \end{cases}$ l. $\begin{cases} x+5y+7z=12 \\ 2x+8y+12z=22 \\ 3x+12y+20z=39 \end{cases}$

5. Solve each system using Cramer's Rule if it is applicable

a. $\begin{cases} 5x-y=13 \\ 2x+3y=12 \end{cases}$ b. $\begin{cases} 3x-6y=24 \\ 5x+4y=12 \end{cases}$ c. $\begin{cases} 3x-2y=4 \\ 6x-4y=0 \end{cases}$

d. $\begin{cases} 2x-4y=-2 \\ 3x+2y=3 \end{cases}$ e. $\begin{cases} 3x-2y=0 \\ 5x+10y=4 \end{cases}$

f. $\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$ g. $\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ -2x + 2y - 3z = 4 \end{cases}$ h. $\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$

6. Solve

a. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 8 \\ \dfrac{3}{x} - \dfrac{5}{y} = 0 \end{cases}$ b. $\begin{cases} \dfrac{4}{x} - \dfrac{3}{y} = 0 \\ \dfrac{6}{x} + \dfrac{3}{2y} = 2 \end{cases}$

7. Solve for x

a. $\begin{vmatrix} x & x \\ 4 & 3 \end{vmatrix} = 5$ b. $\begin{vmatrix} x & 1 \\ 3 & x \end{vmatrix} = -2$ c. $\begin{vmatrix} x & 1 & 1 \\ 4 & 3 & 2 \\ -1 & 2 & 5 \end{vmatrix} = 2$

d. $\begin{vmatrix} x & 1 & 2 \\ 1 & x & 3 \\ 0 & 1 & 2 \end{vmatrix} = -4x$

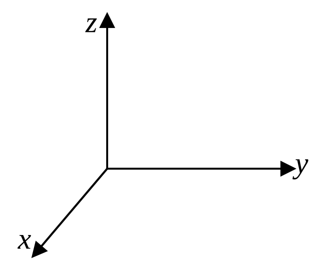
Chapter 4

Vectors

1 Rectangular Coordinates In 3-Space

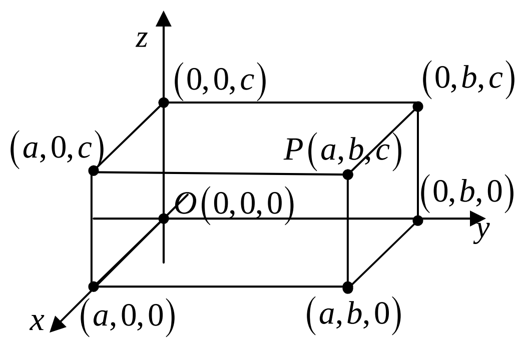
1.1 Rectangular Coordinate Systems

The three coordinate axes form a three-dimensional **rectangular** or **Cartesian coordinate system**, and the point of intersection of the coordinate axes is called the **origin** of the coorrdinate system (see the figure.)



Each pair of coordinate axes determines a plane called a **coordinate plane**. These are referred to as the *xy-plane*, the *xz-plane*, and the *yz-plane*.

To each point P in 3-space we assign a triple numbers (a,b,c) called the coordinates of P.



1.2 Distance Formula In 3-Space

The distance from the origin to the point $P(a,b,c)$ is found by $d = \sqrt{a^2+b^2+c^2}$

The students should derive this formula using the Theorem of Pythagoras.

More generally, the distance from any point $P_1(x_1,y_1,z_1)$ to any other point $P_2(x_2,y_2,z_2)$ is defined by

$$d = \sqrt{|x_2-x_1|^2+|y_2-y_1|^2+|z_2-z_1|^2}$$

or equivalently

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Example 1

The distance between the point $P_1(4,-1,3)$ and the point $P_2(2,3,-1)$ is

$$d = \sqrt{(4-2)^2+(-1-3)^2+(3+1)^2} = \sqrt{36} = 6$$

The mid-point M of the line segment joining the point $P_1(x_1, y_1, z_1)$ and the point $P_2(x_2, y_2, z_2)$ is

$$M\left(\tfrac{1}{2}(x_1+x_2), \tfrac{1}{2}(y_1+y_2), \tfrac{1}{2}(z_1+z_2)\right)$$

Example 2

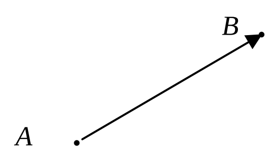
The midpoint of the line segment joining the points $(-1,3,-8)$ and the point $(3,1,0)$ is

$$\left(\tfrac{1}{2}(-1+3), \tfrac{1}{2}(3+1), \tfrac{1}{2}(-8+0)\right) = (1,2,-4)$$

2 Vectors

2.1 Vectors in Geometric View

Vectors can be represented geometrically as directed line segments or arrows in two or three dimensional space; the direction of the arrow specifies the direction of the vector and the length of the arrow describes its magnitude. The tail of the arrow is called the **initial point** of the vector, and the tip of the arrow the **terminal point**. When discussing vectors, we shall refer to real numbers as **scalars**.



If the initial point of a vector $\vec{v}$ is A and the terminal point is B, we write $\vec{v} = \overrightarrow{AB}$

Vectors having the same length and same direction are called **equivalent**. Since we want a vector to be determined solely by its length and direction, equivalent vectors are regarded as **equal** even though they may be located in different positions. If $\vec{v}$ and $\vec{w}$ are equivalent, we write $\vec{v} = \vec{w}$.

Definition

If $\vec{v}$ and $\vec{w}$ are any two vectors, then the sum $\vec{v}+\vec{w}$ is the vector determined as follows. Position the vector $\vec{w}$ so that its initial point coincides with the terminal point of $\vec{v}$. The vector $\vec{v}+\vec{w}$ is represented by the arrow from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$. It is obvious that $\vec{v}+\vec{w} = \vec{w}+\vec{v}$.

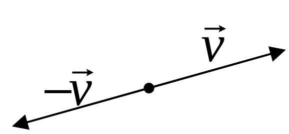
The vector of length zero is called the **zero vector** and is denoted by $\vec{0}$. We defined

$$\vec{0}+\vec{v} = \vec{v}+\vec{0}$$

Note that the direction of any zero vector is undefined.

If $\vec{v}$ is any nonzero vector, then $-\vec{v}$, the **negative** of $\vec{v}$, is defined to be the vector having the same magnitude as $\vec{v}$, but oppositely directed. This vector has the property

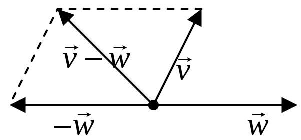
$$\vec{v}+(-\vec{v}) = \vec{0}$$



Definition

If $\vec{v}$ and $\vec{w}$ are any two vectors, then subtraction of $\vec{w}$ from $\vec{v}$ is defined by

$$\vec{v}-\vec{w}=\vec{v}+(-\vec{w})$$



Definition

If $\vec{v}$ is a nonzero vector and k is a nonzero real number (scalar), then the **product** $k\vec{v}$ is defined to be the vector whose length is $|k|$ times the length of $\vec{v}$ and whose direction is the same as that of $\vec{v}$ if $k>0$ and opposite to that of $\vec{v}$ if $k<0$. We define

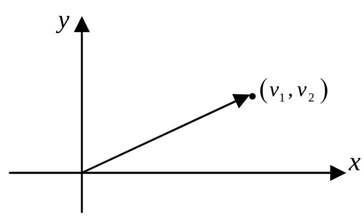
$$k\vec{v}=\vec{0} \text{ if } k=0 \text{ or } \vec{v}=\vec{0}$$

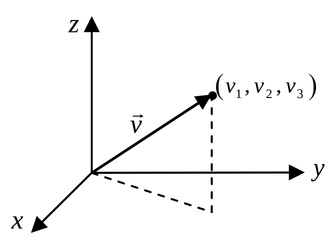
2.2 Vectors in Coordinate Systems

If $\vec{v}$ is a vector in 2-space with its initial point at the origin of a rectangular coordinate system, then the coordinate (v_1,v_2) or (v_1,v_2,v_3) of the terminal point are called **components** of $\vec{v}$ and we write

$$\vec{v}=(v_1,v_2) \text{ or } \vec{v}=(v_1,v_2,v_3)$$

depending on whether the vector is in 2-space or in 3-space.





2.3 Equality of vectors

The vectors $\vec{v}=(v_1,v_2)$ and $\vec{w}=(w_1,w_2)$ in 2-space are equivalent (i.e., $\vec{v}=\vec{w}$) if and only if

$$v_1=w_1 \text{ and } v_2=w_2$$

and vectors

$$\vec{v}=(v_1,v_2,v_3) \text{ and } \vec{w}=(w_1,w_2,w_3)$$

in 3-space are equivalent if and only if

$$v_1=w_1, v_2=w_2 \text{ and } v_3=w_3$$

2.4 Arithmetic Operations on Vectors

If $\vec{v}=(v_1,v_2)$ and $\vec{w}=(w_1,w_2)$ are vectors in 2-space and k is any scalar, then

$$\vec{v}\pm\vec{w}=(v_1\pm w_1, v_2\pm w_2)$$

$$k\vec{v} = (kv_1, kv_2)$$

Similarly, if $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ are vector in 3-space and k is any scalar, then

$$\vec{v} \pm \vec{w} = (v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3)$$
$$k\vec{v} = (kv_1, kv_2, kv_3)$$

Example 1

If $\vec{v} = (1,-2)$ and $\vec{w} = (7,6)$, then find $\vec{v}+\vec{w}$, $4\vec{v}$, $-\vec{v}$, and $\vec{v}-\vec{w}$

Solution

$$\vec{v}+\vec{w} = (1,-2)+(7,6) = (1+7,-2+6) = (8,4)$$
$$4\vec{v} = 4(1,-2) = (4,-8)$$
$$-\vec{v} = (-1)\vec{v} = -1(1,-2) = (-1,2)$$
$$\vec{v}-\vec{w} = (1,-2)-(7,6) = (1-7,-2-6) = (-6,-8)$$

Example 2

Let $\vec{v} = (-2,0,1)$ and $\vec{w} = (3,5,-4)$. Find $\vec{v}+\vec{w}, -3\vec{v}, -\vec{w}$ and $\vec{w}-2\vec{v}$

Solution

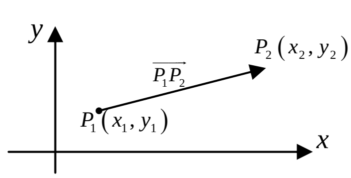
$$\vec{v}+\vec{w} = (-2,0,1)+(3,5,-4) = (1,5,-3)$$
$$-3\vec{v} = (6,0,-3)$$
$$-\vec{w} = (-3,-5,4)$$
$$\vec{w}-2\vec{v} = (3,5,-4)-(-4,0,2) = (7,5,-6)$$

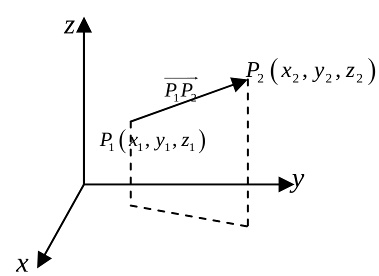
2.5 Vectors with Initial Point Not at the Origin

If $\overrightarrow{P_1P_2}$ is a vector in 2-space with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$,then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

Similarly, if $\overrightarrow{P_1P_2}$ is a vector in 3-space with initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$





(The proof is exercise)

Example 3

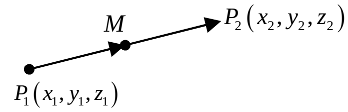
In 2-space the vector with initial point $P_1(1,3)$ and terminal point $P_2(4,-2)$ is

$$\overrightarrow{P_1P_2} = (4-1,-2-3) = (3,-5)$$

and in 3-space the vector with initial point $A(0,-2,5)$ and terminal point $B(3,4,-1)$ is

$$\overrightarrow{AB}=(3-0,4-(-2),-1-5)=(3,6,-6)$$

$$\overrightarrow{P_1M}=\frac{1}{2}\overrightarrow{P_1P_2}$$



2.6 Rules of Vector Arithmetic

For any vectors $\vec{u},\vec{v}$ and $\vec{w}$ and any scalars *k* and *l*, the following relationships hold:

i. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
ii. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
iii. $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$
iv. $\vec{u}+(-\vec{u})=\vec{0}$
v. $k(l\vec{u})=(kl)\vec{u}$
vi. $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$
vii. $(k+l)\vec{u}=k\vec{u}+l\vec{u}$
viii. $1\vec{u}=\vec{u}$

2.7 Length of a Vector

Geometrically, the **length** of a vector $\vec{v}$, also called the **norm** of $\vec{v}$ is the distance between its initial and terminal points. The length (or norm) of $\vec{v}$ is denoted by $\|\vec{v}\|$. It follows from the distance formulas in 2-space and 3-space that the norm of a vector $\vec{v}=(v_1,v_2)$ in 2-space is given by

$$\|\vec{v}\|=\sqrt{v_1^2+v_2^2}$$

and the norm of a vector $\vec{v}=(v_1,v_2,v_3)$ in 3-space is given by

$$\|\vec{v}\|=\sqrt{v_1^2+v_2^2+v_3^2}$$

Example 4

Find the norm of $\vec{v}=(-2,3)$ and $\vec{w}=(2,3,6)$

Solution

$$\|\vec{v}\|=\sqrt{(-2)^2+3^2}=\sqrt{13}$$

$$\|\vec{w}\|=\sqrt{2^2+3^2+6^2}=\sqrt{49}=7$$

Remark The length of vector $k\vec{v}$ is $|k|$ times the length of $\vec{v}$. To express as an equation, we can write as $\|k\vec{v}\|=|k|\|\vec{v}\|$

This formula applies to both vectors in 2-space and 3-space.

2.8 Unit Vector

It follows from the formula $\|k\vec{v}\| = |k|\|\vec{v}\|$ that if a nonzero vector $\vec{v}$ is multiplied by $1/\|\vec{v}\|$ (the reciprocal of its length), then the result is a vector of length 1 in the same direction a $\vec{v}$. This process of multiplying $\vec{v}$ by $1/\|\vec{v}\|$ to obtain a vector of length 1 is called **normalizing** $\vec{v}$.

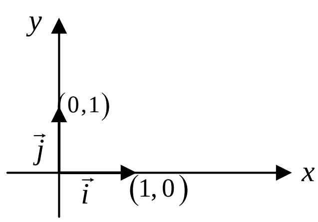
Example 5
A vector of length 1 in the direction of vector $\vec{v} = (3,4)$ is

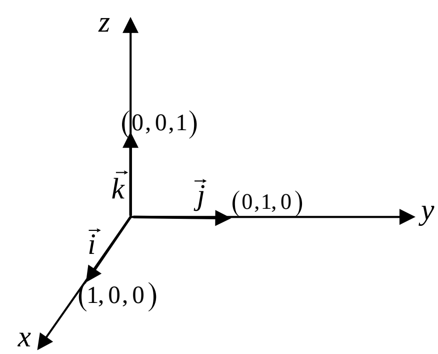
$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{3^2+4^2}}(3,4) = \frac{1}{5}(3,4) = \left(\frac{3}{5},\frac{4}{5}\right)$$

A vector of length 1 is called a **unit vector**. Of special importance are unit vectors that run along the positive coordinate axes of a rectangular coordinate system.
In 2-space the unit vectors along x and y axes are denoted by $\vec{i}$ and $\vec{j}$ respectively. In 3-space the unit vectors along the x, y, and z-axes are $\vec{i}$, $\vec{j}$, and $\vec{k}$. Thus,

$$\vec{i} = (1,0), \qquad \vec{j} = (0,1) \qquad \text{in 2-space}$$

$$\vec{i} = (1,0,0), \qquad \vec{j} = (0,1,0), \qquad \vec{k} = (0,0,1) \qquad \text{in 3-space}$$





Every vector $\vec{v} = (v_1, v_2)$ in 2-space is expressible uniquely in terms of $\vec{i}$ and $\vec{j}$ since we can write

$$\vec{v} = (v_1,v_2) = (v_1,0)+(0,v_2) = v_1(1,0)+v_2(0,1) = v_1\vec{i}+v_2\vec{j}$$

and similarly every vector $\vec{v} = (v_1, v_2, v_3)$ in 3-space is expressible uniquely in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$

$$\vec{v} = (v_1,v_2,v_3) = v_1(1,0,0)+v_2(0,1,0)+v_3(0,0,1) = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$$

Remark The notations (v_1,v_2,v_3) and $v_i\vec{i}+v_2\vec{j}+v_3\vec{k}$ are interchangeable and we use both of them. Similarly, the notations (v_1,v_2) and $v_1\vec{i}+v_2\vec{j}$ are interchangeable.

Example 6:

$$(2,3) = 2\vec{i}+3\vec{j} \qquad (0,0) = 0\vec{i}+0\vec{j} = \vec{0} \qquad \|2\vec{i}-3\vec{j}\| = \sqrt{2^2+(-3)^2} = \sqrt{13}$$

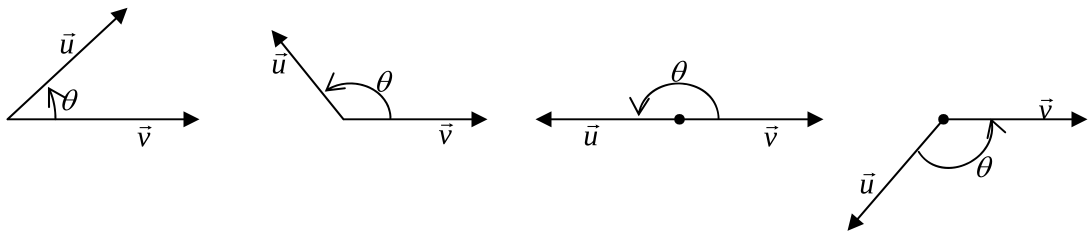
$$(2,-3,4) = 2\vec{i}-3\vec{j}+4\vec{k} \qquad (0,3,0) = 3\vec{j}$$

$$\|\vec{i}+2\vec{j}-3\vec{k}\| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$$

3 Dot Product; Projections

3.1 Angle between Vectors

Let $\vec{u}$ and $\vec{v}$ be two nonzero vectors in 2-space or 3-space, and assume these vectors have been positioned so that their initial points coincide. By the *angle between* $\vec{u}$ *and* $\vec{v}$ we shall mean the angle θ determined by $\vec{u}$ and $\vec{v}$ that satisfies $0 \le \theta \le \pi$.



Definition

If $\vec{u}$ and $\vec{v}$ are vectors in 2-space or 3-space and θ is the angle between $\vec{u}$ and $\vec{v}$, then the **dot product** or **Euclidean inner product** $\vec{u}\cdot\vec{v}$ is defined by

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

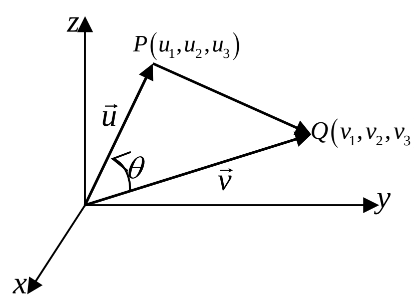
Example 1

The angle between vector $\vec{u}=(0,2)$ and $\vec{v}=(1,1)$ is 45° . Thus,

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta = \sqrt{0^2+2^2}\sqrt{1^2+1^2}\cos 45^\circ = 2\times\sqrt{2}\times\frac{1}{\sqrt{2}} = 2$$

3.2 Formular for the Dot Product

Let $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ be two nonzero vectors. If θ is the angle between $\vec{u}$ and $\vec{v}$, then the law of cosines yields



$$\left\|\overrightarrow{PQ}\right\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Since $\overrightarrow{PQ} = \vec{v}-\vec{u}$ we can obtain

$$\|\vec{u}\|\|\vec{v}\|\cos\theta = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v}-\vec{u}\|^2\right)$$

or

$$\vec{u}\cdot\vec{v} = \frac{1}{2}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v}-\vec{u}\|^2\right)$$

substituting

$$\|\vec{u}\|^2 = u_1^2+u_2^2+u_3^2, \|\vec{v}\|^2 = v_1^2+v_2^2+v_3^2$$

and

$$\|\vec{v}-\vec{u}\|^2 = (v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2$$

we obtain, after simplifying

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

If $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$ are two vectors in 2-space, then the formula is

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2$$

From the definition, if $\vec{u}$ and $\vec{v}$ are nonzero vectors, then we can write

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Example 2

Consider the vectors $\vec{u} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{v} = \vec{i} + \vec{j} + 2\vec{k}$. Find $\vec{u}\cdot\vec{v}$ and determine the angle θ between $\vec{u}$ and $\vec{v}$.

Solution

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = 2\times1 + (-1)\times1 + 1\times2 = 3$$

$$\|\vec{u}\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6} \text{ and } \|\vec{v}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

so that

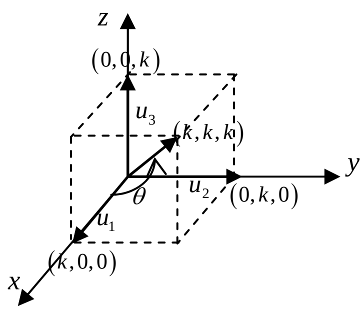
$$\cos\theta = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

Thus, $\theta = \pi/3$

Example 3

Find the angle between a diagonal of a cube and one of its edges.

Solution



Let k be the length of an angle and let introduce a coordinate system as in the figure.

If we let $\vec{u}_1 = (k,0,0), \vec{u}_2 = (0,k,0)$ and $\vec{u}_3 = (0,0,k)$ then the vector $\vec{d} = \vec{u}_1 + \vec{u}_2 + \vec{u}_3$ is a diagonal of a cube. The angle θ between $\vec{d}$ and the edge $\vec{u}_1$ satisfies

$$\cos\theta = \frac{\vec{u}_1\cdot\vec{d}}{\|\vec{u}_1\|\|\vec{d}\|} = \frac{k^2}{k\times\sqrt{3k^2}} = \frac{1}{\sqrt{3}}$$

Thus, $\theta = \arccos\frac{1}{\sqrt{3}}$.

Remark If $\vec{u}$ and $\vec{v}$ are nonzero vectors in 2-space or 3-space and if θ is the angle between them, then

i. θ is acute if and only if $\vec{u}\cdot\vec{v} > 0$

ii. θ is obtuse if and only if $\vec{u}\cdot\vec{v} < 0$

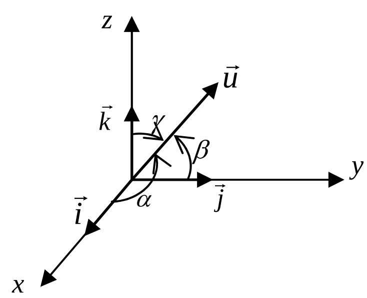
iii. $\theta = \frac{\pi}{2}$ if and only if $\vec{u}\cdot\vec{v} = 0$

3.3 Orthogonal Vectors

Perpendicular vectors are also called **orthogonal vectors**. From the remark above, two nonzero vectors are orthogonal if and only if their dot product is zero. If we agree to consider $\vec{u}$ and $\vec{v}$ to be perpendicular when either or both of these vectors is $\vec{0}$. So, we can state that **two vectors $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) if and only if $\vec{u}\cdot\vec{v} = \vec{0}$.**

3.4 Direction Cosines

Of special interest are the angel $\alpha, \beta,$ and γ that a vector $\vec{u}$ in 3-space makes with the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. These are called **direction angles** of $\vec{u}$. The numbers $\cos\alpha, \cos\beta,$ and $\cos\gamma$ are called the **direction cosines** of $\vec{u}$.



The three direction cosines of a nonzero vector $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ in 3-space are

$$\cos\alpha = \frac{u_1}{\|\vec{u}\|}, \cos\beta = \frac{u_2}{\|\vec{u}\|}, \text{and } \cos\gamma = \frac{u_3}{\|\vec{u}\|}$$

The proof is considered exercise.

Example 4

Find the direction cosines of the vector $\vec{u} = 2\vec{i} - 4\vec{j} + 4\vec{k}$ and estimate the direction angles.

Solution

$$\|\vec{u}\| = \sqrt{4+16+16} = 6$$

$$\cos\alpha = \frac{2}{6} = \frac{1}{3} \quad \cos\beta = \frac{-4}{6} = \frac{-2}{3}, \text{ and } \cos\gamma = \frac{4}{6} = \frac{2}{3}$$

Hence, $\alpha = \arccos\frac{1}{3}, \beta = \arccos\left(-\frac{2}{3}\right)$ and $\gamma = \arccos\frac{2}{3}$

3.5 Properties of the Dot Product

If $\vec{u}, \vec{v},$ and $\vec{w}$ are vectors in 2- or 3-space and k is a scalar, then

i. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
ii. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
iii. $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})$
iv. $\vec{v}\cdot\vec{v} = \|v\|^2$

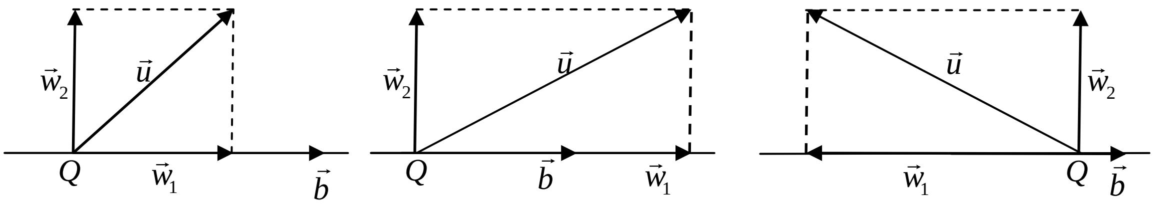
3.6 Orthogonal Projections of Vectors

In many applications, it is of interest to decompose a vector $\vec{u}$ into a sum of to terms, one parallel to a specified nonzero vector $\vec{b}$ and the other perpendicular to $\vec{b}$. If $\vec{u}$ and $\vec{b}$ are positioned so that their initial points coincide at a point Q, we may decompose the vector $\vec{u}$ as follows(see the figure): Drop a perpendicular from the tip of $\vec{u}$ to the line through $\vec{b}$ and construct the vector $\vec{w}_1$ from Q to the foot of the perpendicular; next from the difference

$$\vec{w}_2 = \vec{u} - \vec{w}_1$$

The vector $\vec{w}_1$ is parallel to $\vec{b}$, the vector $\vec{w}_2$ is perpendicular to $\vec{b}$ and

$$\vec{w}_1 + \vec{w}_2 = \vec{w}_1 + (\vec{u} - \vec{w}_1) = \vec{u}$$



$\vec{u} = \vec{w}_1 + \vec{w}_2$

The vector $\vec{w}_1$ is called the **orthogonal projection of $\vec{u}$ on $\vec{b}$** or sometimes the **vector component of $\vec{u}$ along $\vec{b}$**. It is denoted by $proj_{\vec{b}}\vec{u}$.

The vector $\vec{w}_2$ is called the **vector component of $\vec{u}$ orthogonal to $\vec{b}$**. Since $\vec{w}_2 = \vec{u} - \vec{w}_1$, this vector may be written as $\vec{w}_2 = \vec{u} - proj_{\vec{b}}\vec{u}$.

Formula for calculating

If $\vec{u}$ and $\vec{b}$ are vectors in 2-space or 3-space and if $\vec{b} \neq \vec{0}$, then

$$proj_{\vec{b}}\vec{u} = \frac{\vec{u}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} \text{ (Vector component of } \vec{u} \text{ along } \vec{b}\text{)}$$

$$\vec{u} - proj_{\vec{b}}\vec{u} = \vec{u} - \frac{\vec{u}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} \text{ (Vector component of } \vec{u} \text{ orthogonal to } \vec{b}\text{)}$$

Example 3

Let $\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}$ and $\vec{b} = 4\vec{i} - \vec{j} + 2\vec{k}$. Find the vector component of $\vec{u}$ along $\vec{b}$ and the vector component of $\vec{u}$ orthogonal to $\vec{b}$.

Solution

$$\vec{u}\cdot\vec{b} = 2\times 4 + (-1)(-1) + 3\times 2 = 15$$

$$\|\vec{b}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus the vector component of $\vec{u}$ along $\vec{b}$ is

$$proj_{\vec{b}}\vec{u} = \frac{\vec{u}\cdot\vec{b}}{\|b\|^2}\vec{b} = \frac{15}{21}\left(4\vec{i} - \vec{j} + 2\vec{k}\right) = \frac{20}{7}\vec{i} - \frac{5}{7}\vec{j} + \frac{10}{7}\vec{k}$$

and the vector component of $\vec{u}$ orthogonal to $\vec{b}$ is

$$\vec{u} - proj_{\vec{b}}\vec{u} = \left(2\vec{i} - \vec{j} + 3\vec{k}\right) - \left(\frac{20}{7}\vec{i} - \frac{5}{7}\vec{j} + \frac{10}{7}\vec{k}\right)$$

$$= -\frac{6}{7}\vec{i} - \frac{2}{7}\vec{j} + \frac{11}{7}\vec{k}$$

A formula for the length of the vector component of $\vec{u}$ along $\vec{b}$ may be obtained by writing

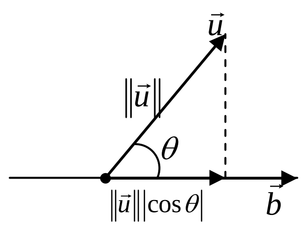
$$\left\|proj_{\vec{b}}\vec{u}\right\| = \left\|\frac{\vec{u}\cdot\vec{b}}{\left\|\vec{b}\right\|^2}\vec{b}\right\| = \left|\frac{\vec{u}\cdot\vec{b}}{\left\|\vec{b}\right\|^2}\right|\left\|\vec{b}\right\|$$

$$= \frac{\left|\vec{u}\cdot\vec{b}\right|}{\left\|\vec{b}\right\|^2}\left\|\vec{b}\right\| = \frac{\left|\vec{u}\cdot\vec{b}\right|}{\left\|\vec{b}\right\|}$$

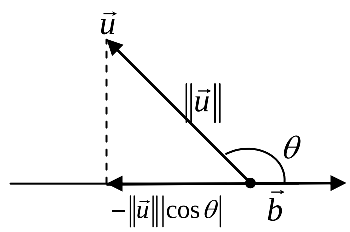
Hence, $\left\|proj_{\vec{b}}\vec{u}\right\| = \dfrac{\left|\vec{u}\cdot\vec{b}\right|}{\left\|\vec{b}\right\|}$

If θ denotes the angle between $\vec{u}$ and $\vec{b}$, then $\vec{u}\cdot\vec{b} = \left\|\vec{u}\right\|\left\|\vec{b}\right\|\cos\theta$, so we obtain

$$\left\|proj_{\vec{b}}\vec{u}\right\| = \left\|\vec{u}\right\|\left|\cos\theta\right|$$



$0 \le \theta < \dfrac{\pi}{2}$



$\dfrac{\pi}{2} < \theta \le \pi$

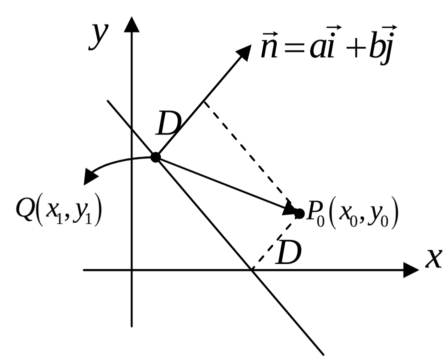
Example 4

Derive the formula for the distance D between the point $P_0\left(x_0, y_0\right)$ and the line $ax + by + c = 0$.

Solution

From the figure, the distance D is equal to the length of the orthogonal projection of the vector $\overrightarrow{QP_0}$ on $\vec{n}$; thus,

$$D = \left\|proj_{\vec{n}}\overrightarrow{QP_0}\right\| = \frac{\left|\overrightarrow{QP_0}\cdot\vec{n}\right|}{\left\|\vec{n}\right\|}$$



But

$$\overrightarrow{QP_0} = \left(x_0 - x_1, y_0 - y_1\right)$$

$$\overrightarrow{QP_0}\cdot\vec{n} = a\left(x_0 - x_1\right) + b\left(y_0 - y_1\right)$$

$$\left\|\vec{n}\right\| = \sqrt{a^2 + b^2}$$

So

$$D = \frac{\left|a\left(x_0 - x_1\right) + b\left(y_0 - y_1\right)\right|}{\sqrt{a^2 + b^2}}$$

since the point $Q\left(x_1, y_1\right)$ lies on the line, its coordinates satisfy the equation of the line, so

$$ax_1 + by_1 + c = 0 \text{ or } c = -ax_1 - by_1$$

Hence, we obtain

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

4 Cross Product

4.1 Cross Product

Definition

If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the **cross product** $\vec{u} \times \vec{v}$ is the vector defined by

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

or

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example 1

Find $\vec{u} \times \vec{v}$ where $\vec{u} = (1, 2, -2)$ and $\vec{v} = (3, 0, 1)$

Solution

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \vec{k} = 2\vec{i} - 7\vec{j} - 6\vec{k}$$

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then

i. $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{0}$ ($\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$)

ii. $\vec{v} \cdot (\vec{u} \times \vec{v}) = \vec{0}$ ($\vec{u} \times \vec{v}$ is orthogonal to $\vec{v}$)

iii. $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$ (Lagrange's identity)

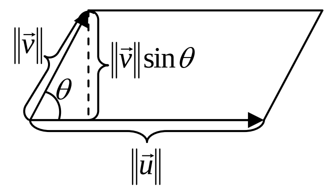
The proof is an exercise.

Example 2

Prove that $\vec{u} = (1, 2, -2)$ and $\vec{v} = (3, 0, 1)$ satisfy i. and ii.

Lagrange's identity in the theorem states that $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$. If θ denotes the angle between $\vec{u}$ and $\vec{v}$, then $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$, so we can write

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - \|\vec{u}\|^2\|\vec{v}\|^2\cos^2\theta$$
$$= \|\vec{u}\|^2\|\vec{v}\|^2\left(1-\cos^2\theta\right)$$
$$= \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta$$



Since $0\le\theta\le\pi$, it follows that $\sin\theta\ge 0$, so
$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$$
But $\|\vec{v}\|\sin\theta$ is the altitude of the parallelogram determined by $\vec{u}$ and $\vec{v}$. Thus, the area A of the parallelogram is given by
$$A = base\times altitude = \|\vec{u}\|\|\vec{v}\|\sin\theta = \|\vec{u}\times\vec{v}\|$$

Example 3

Find area A of the triangle that is determined by the points $P_1(2,2,0), P_2(-1,0,2)$ and $P_3(0,4,3)$ (ans: 15/2)

Properties If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\vec{u}\times\vec{v}=\vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

If $\vec{u},\vec{v},$ and $\vec{w}$ are any vectors in 3-space and k is any scalar, then

i. $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

ii. $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

iii. $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$

iv. $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$

v. $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$

vi. $\vec{u}\times\vec{u} = \vec{0}$

Cross products of the unit vectors $\vec{i}, \vec{j},$ and $\vec{k}$ are of special interest. For example,
$$\vec{i}\times\vec{j} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix} = \begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}\vec{i} - \begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}\vec{j} + \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\vec{k} = \vec{k}$$

Other cross products are listed below

$\vec{i}\times\vec{j} = \vec{k}$ $\quad$ $\vec{j}\times\vec{k} = \vec{i}$ $\quad$ $\vec{k}\times\vec{i} = \vec{j}$

$\vec{j}\times\vec{i} = -\vec{k}$ $\quad$ $\vec{k}\times\vec{j} = -\vec{i}$ $\quad$ $\vec{i}\times\vec{k} = -\vec{j}$

$\vec{i}\times\vec{i} = \vec{0}$ $\quad$ $\vec{j}\times\vec{j} = \vec{0}$ $\quad$ $\vec{k}\times\vec{k} = \vec{0}$

4.2 Triple Scalar Products

If $\vec{a}=(a_1,a_2,a_3), \vec{b}=(b_1,b_2,b_3)$ and $\vec{c}=(c_1,c_2,c_3)$ are vector in 3-space, then the number
$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$$
are called **triple scalar product** of $a, \vec{b}$ and $\vec{c}$. It is calculated from the formula

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This formula derives from

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{a}\cdot\left(\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right)=\vec{a}\cdot\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\vec{i}-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\vec{j}+\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\vec{k}\right)$$

$$=\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}a_1-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}a_2+\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}a_3=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 4

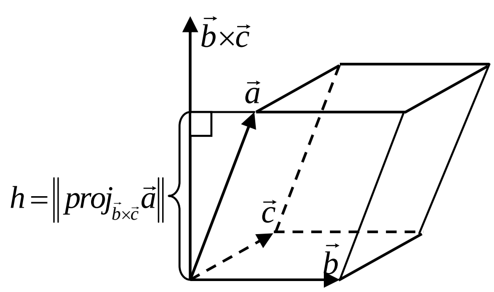
Calculate the triple scalar product $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ of the vector $\vec{a}=\left(3\vec{i}-2\vec{j}-5\vec{k}\right)$, $\vec{b}=\left(\vec{i}+4\vec{j}-4\vec{k}\right)$ and $\vec{c}=\left(3\vec{j}+2\vec{k}\right)$

Solution

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}=49$$

Remark We can prove that $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)=\vec{b}\cdot\left(\vec{c}\times\vec{a}\right)$

The triple scalar product $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ has a useful geometric interpretation. If we assume that the vector $\vec{a},\vec{b},$ and $\vec{c}$ do not all lies in the same plane when they are positioned with a common initial point, then the three vectors form adjacent sides of a parallelepiped. If the parallelogram determined by $\vec{b}$ and $\vec{c}$ is regarded as the base of the parallelepiped, then the area of the base is $\left\|\vec{b}\times\vec{c}\right\|$, and the height h is the length of the orthogonal projection of $\vec{a}$ on $\vec{b}\times\vec{c}$ (see the figure). Therefore by the formula for triple scalar product, we have



$$h=\left\|proj_{\vec{b}\times\vec{c}}\vec{a}\right\|=\frac{\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|}{\left\|\vec{b}\times\vec{c}\right\|}$$

It follows that the volume V of the parallelepiped is

$$V=\text{area of base }\times\text{height}=\left\|\vec{b}\times\vec{c}\right\|\frac{\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|}{\left\|\vec{b}\times\vec{c}\right\|}=\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|$$

Remark It follows from this formula that

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\pm V$$

where + or – signs results depending on whether $\vec{a}$ makes an acute or obtuse angle with $\vec{b}\times\vec{c}$.

If the vector $\vec{a}=(a_1,a_2,a_3),\vec{b}=(b_1,b_2,b_3)$ and $\vec{c}=(c_1,c_2,c_3)$ have the same initial point, then they lie in a plane if and only if

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=0$$

Exercises

1. Let $\vec{u}=(1,3),\vec{v}=(2,1)$ and $\vec{w}=(4,-1)$. Find

a. $\vec{u}-\vec{w}$ b. $7\vec{v}+3\vec{w}$ c. $-\vec{w}+\vec{v}$ d. $3(\vec{u}-7\vec{v})$

e. $-3\vec{v}-8\vec{w}$ f. $2\vec{v}-(\vec{u}+\vec{w})$

2. Let $\vec{u}=2\vec{i}+3\vec{j},\vec{v}=\vec{i}$ and $\vec{w}=-\vec{i}-2\vec{j}$. Find

a. $\vec{w}-\vec{v}$ b. $6\vec{u}+4\vec{w}$ c. $-\vec{v}-2\vec{w}$ d. $4(3\vec{u}+\vec{v})$

e. $-8(\vec{v}+\vec{w})+2\vec{u}$ f. $3\vec{w}-(\vec{v}-\vec{w})$

3. Let $\vec{u}=(2,-1,3),\vec{v}=(4,0,-2),\vec{w}=(1,1,3)$. Find

a. $\vec{u}-\vec{w}$ b. $7\vec{v}+3\vec{w}$ c. $-\vec{w}+\vec{v}$ d. $3(\vec{u}-7\vec{v})$

e. $-3\vec{v}-8\vec{w}$ f. $2\vec{v}-(\vec{u}+\vec{w})$

4. Let $\vec{u}=\left(3\vec{i}-\vec{k}\right),\vec{v}=\vec{i}-\vec{j}+2\vec{k},\vec{w}=3\vec{j}$. Find

a. $\vec{w}-\vec{v}$ b. $6\vec{u}+4\vec{w}$ c. $-\vec{v}-2\vec{w}$ d. $4(3\vec{u}+\vec{v})$

e. $-8(\vec{v}+\vec{w})+2\vec{u}$ f. $3\vec{w}-(\vec{v}-\vec{w})$

In Exercises 5-8, compute the norm of $\vec{v}$.

5. a. $\vec{v}=(3,4)$ b. $\vec{v}=-\vec{i}+7\vec{j}$ c. $\vec{v}=-3\vec{j}$

6. a. $\vec{v}(1,-1)$ b. $\vec{v}=(2,0)$ c. $\vec{v}=\sqrt{2}\vec{i}-\sqrt{7}\vec{j}$

7. a. $\vec{v}=\vec{i}+\vec{j}+\vec{k}$ b. $\vec{v}=(-1,2,4)$

8. a. $\vec{v}=-3\vec{i}+2\vec{j}+\vec{k}$ b. $\vec{v}=(0,-3,0)$

9. Let $\vec{u}=(1,-3)$, $\vec{v}(1,1)$ and $\vec{w}=(2,-4)$. Find

a. $\|\vec{u}+\vec{v}\|$ b. $\|\vec{u}\|+\|\vec{v}\|$ c. $\|-2\vec{u}\|+2\|\vec{v}\|$

d. $\|3\vec{u}-5\vec{v}+\vec{w}\|$

10. Let $\vec{u}=2\vec{i}-5\vec{j}, \vec{v}=2\vec{i}, \vec{w}=3\vec{i}+4\vec{j}$. Find

a. $\|\vec{v}+\vec{w}\|$ b. $\|\vec{v}\|+\|\vec{w}\|$ c. $\|-3\vec{u}\|+4\|\vec{v}\|$

d. $\|\vec{u}-\vec{v}-\vec{w}\|$ e. $\frac{1}{\|\vec{w}\|}\vec{w}$ f. $\left\|\frac{1}{\|\vec{w}\|}\vec{w}\right\|$

11. Let $\vec{u}=(2,-1,0), \vec{v}=(0,1,-1)$. Find

a. $\|\vec{u}+\vec{v}\|$ b. $\|\vec{u}\|+\|\vec{v}\|$ c. $\|3\vec{u}\|$ d. $\|2\vec{u}-3\vec{v}\|$

12. Let $\vec{u}=\vec{i}-3\vec{j}+2\vec{k}, \vec{v}=\vec{i}+\vec{j}$ and $\vec{w}=2\vec{i}+2\vec{j}-4\vec{k}$. Find

a. $\|\vec{u}+\vec{v}\|$ b. $\|\vec{u}\|+\|\vec{v}\|$ c. $\|-2\vec{u}\|+2\|\vec{v}\|$

d. $\|3\vec{u}-5\vec{v}+\vec{w}\|$ e. $\frac{1}{\|\vec{w}\|}\vec{w}$ f. $\left\|\frac{1}{\|\vec{w}\|}\right\|\vec{w}$

13. Let $\vec{u}=(-1,1), \vec{v}=(0,1)$ and $\vec{w}=(3,4)$. Find the vector $\vec{x}$ that satisfies $\vec{u}-2x=x-\vec{w}+3\vec{v}$.

14. Let $\vec{u}=(1,3), \vec{v}=(2,1), \vec{w}=(3,4)$. Find the vector $\vec{x}$ that satisfies $2\vec{u}-\vec{v}+\vec{x}=7\vec{x}+\vec{w}$.

15. Find $\vec{u}$ and $\vec{v}$ if $\vec{u}+\vec{v}=(2,-3)$ and $3\vec{u}+2\vec{v}=(-1,2)$.

16. Find $\vec{u}$ and $\vec{v}$ if $\vec{u}+2\vec{v}=3\vec{i}+\vec{k}$ and $3\vec{u}-\vec{v}=\vec{i}+\vec{j}+\vec{k}$.

17. Let $\vec{u}=2\vec{i}-\vec{j}$ and $\vec{v}=4\vec{i}+2\vec{j}$. Find scalars c_1 and c_2 such that $c_1\vec{u}+c_2\vec{v}=-4\vec{j}$.

18. Let $\vec{u}=(1,-3)$ and $\vec{v}=(-2,6)$. Show that there do not exist scalars c_1 and c_2 such that $c_1\vec{u}+c_2\vec{v}=(3,5)$.

19. Let $\vec{u}=(1,0,1), \vec{v}=(3,2,0)$ and $\vec{w}=(0,1,1)$. Find scalars $c_1, c_2,$ and c_3 such that $c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=(-1,1,5)$.

20. Let $\vec{u}=\vec{i}-\vec{j}, \vec{v}=3\vec{i}+\vec{k}$ and $\vec{w}=4\vec{i}-\vec{j}+\vec{k}$. Show that there do not exist scalars c_1, c_2 and c_3 suchn that $c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=2\vec{i}+\vec{j}-\vec{k}$.

21. Let $\vec{v}=4\vec{i}-3\vec{j}$. Find all scalars k such that $\|k\vec{v}\|=3$.

22. Find a unit vector having the same direction as $-\vec{i}+4\vec{j}$.

23. Find a unit vector having the same direction as $3\vec{i}-4\vec{j}$.

24. Find a unit vector having the same direction as $2\vec{i}-\vec{j}+2\vec{k}$.

25. Find a unit vector having the same director as the vector from the point $A(-3,2)$ to the point $B(1,-1)$.

26. Find a unit vector having the same direction as the vector from the point $A(-1,0,2)$ to the point $B(3,1,1)$.

27. Find a vector having the same direction as the vector $\vec{v}=-2\vec{i}+3\vec{j}$ but with three time the length of $\vec{v}$.

28. Find a vector oppositely directed to $\vec{v}=(3,-4)$ but with halfn of length of $\vec{v}$.

29. Find a vector with the same directed to $\vec{v}=-3\vec{i}+4\vec{j}+\vec{k}$ but with twice the length of $\vec{v}$.

30. Find $\vec{u}.\vec{v}$

a. $\vec{u}=i+2j, \vec{v}=6i-8j$ b. $\vec{u}=(-7,-3), \vec{v}=(0,1)$

c. $\vec{u}=i-3j+7k, \vec{v}=8i-2j-2k$ d. $\vec{u}=(-3,1,2), \vec{v}=(4,2,-5)$

31. In each part of exercise 30, find the cosine of the angle θ between $\vec{u}$ and $\vec{v}$.

32. Determine whether $\vec{u}$ and $\vec{v}$ make an acute angle, an obtuse angle, or are orthogonal .

a. $\vec{u}=7\vec{i}+3\vec{j}+5\vec{k}$, $\vec{v}=-8\vec{i}+8\vec{j}+2\vec{k}$ b. $\vec{u}=6\vec{i}+\vec{j}+3\vec{k}, \vec{v}=4\vec{i}-6\vec{k}$

c. $\vec{u}=(1,1,1), \vec{v}=(-1,0,0)$ d. $\vec{u}=(4,1,6), \vec{v}=(-3,0,2)$

33. Find the orthogonal projection of $\vec{u}$ on $\vec{a}$.

a. $\vec{u}=2\vec{i}+\vec{j}, \vec{a}=-3\vec{i}+2\vec{j}$ b. $\vec{u}=(2,6), \vec{a}=(-9,3)$

c. $\vec{u}=-7\vec{i}+\vec{j}+3\vec{k}, \vec{a}=5\vec{i}+\vec{k}$ d. $\vec{u}=(0,0,1), \vec{a}=(8,3,4)$.

34. In each part of exercise 33, find the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

35. Find $\|\text{proj}_{\vec{a}}\vec{u}\|$

a. $\vec{u}=2\vec{i}-\vec{j}, \vec{a}=3\vec{i}+4\vec{j}$ b. $\vec{u}=(4,5), \vec{a}=(1,-2)$

c. $\vec{u}=2\vec{i}-\vec{j}+3\vec{k}, \vec{a}=\vec{i}+2\vec{j}+2\vec{k}$ d. $\vec{u}=(4,-1,7), \vec{a}=(2,3,-6)$

36. Let $\vec{u}(1,2), \vec{v}(4,-2)$, and $\vec{w}(6,0)$ .Find

a. $\vec{u}(7\vec{v}+\vec{w})$ b. $\|(\vec{u}.\vec{w})\vec{w}\|$ c. $\|\vec{u}\|(\vec{v}.\vec{w})$ d. $(\|\vec{u}\|\vec{v}).\vec{w}$

37. Show that $A(2,-1,1), B(3,2,-1)$ and $C(7,0,-2)$ are vertices of a right triangle. At which vertex is the right angle?

38. Let $\vec{a}=k\vec{i}+\vec{j}$ and $\vec{b}=4\vec{i}+3\vec{j}$. Find k so that

a. $\vec{a}$ and $\vec{b}$ are orthogonal.

b. The angle between $\vec{a}$ and $\vec{b}$ is $\pi/4$

c. The angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$

d. $\vec{a}$ and $\vec{b}$ are parallel.

39. Find the direction cosines of $\vec{u}$ and estimate the direction angles to the nearest degree.

a. $\vec{u}=\vec{i}+\vec{j}-\vec{k}$ b. $\vec{u}=2\vec{i}-2\vec{j}+\vec{k}$

c. $\vec{u}=3\vec{i}-2\vec{j}-6\vec{k}$ d. $\vec{u}=3\vec{i}-4\vec{k}$

40. Calculate the distance between the point and the line.

a. $3x+4y+7=0;(1,-2)$ b. $y=-2x+1;(-3,5)$ c. $2x+y=8;(2,6)$

41. Given the points $A(1,1,0), B(-2,3,-4)$ and $P(-3,1,2)$

a. find $\left\| proj_{A\vec{B}} A\vec{P} \right\|$

b. Use the Pythagorean Theorem and the result of part (a) to find the distance from *P* to the line through *A* and *B*.

42. Find $\vec{a}\times\vec{b}$

a. $\vec{a}=(1,2,-3),\vec{b}=(-4,1,2)$ b. $\vec{a}=3\vec{i}+2\vec{j}-\vec{k},\ \vec{b}=-\vec{i}-3\vec{j}+\vec{k}$

c. $\vec{a}=(0,1,-2),\vec{b}=(3,0,-4)$ d. $\vec{a}=4\vec{i}+\vec{k},\ \vec{b}=2\vec{i}-\vec{j}$

43. Let $\vec{u}=(2,-1,3),\vec{v}=(0,1,7)$ and $\vec{w}=(1,4,5)$. Find

a. $\vec{u}\times(\vec{v}\times\vec{w})$ b. $(\vec{u}\times\vec{v})\times\vec{w}$ c. $\vec{u}\times(\vec{v}-2\vec{w})$ d. $(\vec{u}\times\vec{v})-2\vec{w}$

e. $(\vec{u}\times\vec{v})\times(\vec{v}\times\vec{w})$ f. $(\vec{v}\times\vec{w})\times(\vec{u}\times\vec{v})$

44. Find a vector orthogonal to both $\vec{u}$ and $\vec{v}$.

a. $\vec{u}=-7\vec{i}+3\vec{j}+\vec{k},\vec{v}=2\vec{i}+4\vec{k}$ b. $\vec{u}=(-1,-1,-1),\vec{v}=(2,0,2)$

45. Find the area of the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$.

a. $\vec{u}=\vec{i}-\vec{j}+2\vec{k},\vec{v}=3\vec{j}+\vec{k}$ b. $\vec{u}=2\vec{i}+3\vec{j},\vec{v}=-\vec{i}+2\vec{j}-2\vec{k}$

46. Find the area of the triangle having vertices *P, Q,* and *R*.

a. $P(1,5,-2),Q(0,0,0),R(3,5,1)$ b. $P(2,0,-3),Q(1,4,5),R(7,2,9)$

47. Find $\vec{a}\cdot(\vec{b}\times\vec{c})$

a. $\vec{a}=(1,-2,2),\vec{b}=(0,3,2),\vec{c}=(-4,1,-3)$

b. $\vec{a}=2\vec{i}-3\vec{j}+\vec{k},\ \vec{b}=4\vec{i}+\vec{j}-3\vec{k},\ \vec{c}=\vec{j}+5\vec{k}$

c. $\vec{a}=(2,1,0),\vec{b}=(1,-3,1),\vec{c}=(4,0,1)$

d. $\vec{a}=\vec{i},\vec{b}=\vec{i}+\vec{j},\vec{c}=\vec{i}+\vec{j}+\vec{k}$

48. Find the volume of the parallelepiped with sides $\vec{a},\vec{b},$ and $\vec{c}$

a. $\vec{a}=(2,-6,2),\vec{b}=(0,4,-2),\vec{c}=(2,2,-4)$

b. $\vec{a}=3\vec{i}+\vec{j}+2\vec{k},\vec{b}=4\vec{i}+5\vec{j}+\vec{k},\vec{c}=\vec{i}+2\vec{j}+4\vec{k}$

49. Consider the parallelepiped with sides

$$\vec{a}=3\vec{i}+2\vec{j}+\vec{k}$$
$$\vec{b}=\vec{i}+\vec{j}+2\vec{k}$$
$$\vec{c}=\vec{i}+3\vec{j}+3\vec{k}$$

a. Find the volume

b. Find the area of the face determined by a $\vec{a}$ and $\vec{b}$.

Chapter 5

Complex Numbers

1 The Construction of Complex Numbers

The complex number system will be defined as an extension of the real number system, just as the real number system is an extension of the rational number system. For example, the quadratic equation $x^2 = 2$ has no rational solution, and so a larger system, called the real number system, is introduced, in which this equation does have a solution. The symbol $\sqrt{2}$ is defined as the positive real number whose square is 2.

However, the real number system is not sufficient to solve all quadratic equations. The equation $x^2 = -1$ has no real solution, because the square of any real number can never be negative. We therefore introduce the symbol i to stand for a new kind of number whose square is $x^2 = -1$; that is, $i^2 = -1$. Therefore, we construct all the complex numbers as follows.

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers. The complex number set, which is denoted by the symbol $\mathbb{C}$, is $\{a + bi \mid a, b \in \mathbb{R}\}$

If $z = a + bi$ is such a complex number, then $a + bi$ is said to be the **standard form** of z. The real number a is called the **real part** of z and is denoted by $\operatorname{Re}(z)$. The real number b is called the **imaginary part** of z and is denoted by $\operatorname{Im}(z)$.

For example, $z = 3 + 4i$ is a complex number whose real part is 3 and whose imaginary part is 4. If the imaginary part of a complex number is zero, we equate that complex number with its real part, so that $-6 + 0i$ would be equated with the real number -6. Hence every real number is also a complex number. A complex number whose real part is zero, such as $0 + 5i$ is called **purely imaginary** and is usually denoted by just $5i$.

2 Operations on Complex Numbers

Let z_1 and z_2 be two complex numbers such that $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then

i. Addition: $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$

ii. Subtraction: $z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2) i$

iii. Multiplication:

$$\begin{aligned} z_1 \times z_2 &= (a_1 + b_1 i) \times (a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + b_1 i a_2 + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i \end{aligned}$$

iv. Division:

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

$$= \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i}$$

$$= \frac{\left(a_1 a_2 + b_1 b_2\right) + \left(a_2 b_1 - a_1 b_2\right) i}{a_2^2 + b_2^2}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} i$$

Example 1

If $z = 4 + 7i$ and $w = -3 + i$, find $z + w, z - w$ and zw.

Solution

$$z + w = \left(4 + 7i\right) + \left(-3 + i\right)$$
$$= \left(4 - 3\right) + \left(7 + 1\right) i$$
$$= 1 + 8i$$
$$z - w = \left(4 + 7i\right) - \left(-3 + i\right)$$
$$= \left(4 + 3\right) + \left(7 - 1\right) i$$
$$= 7 + 6i$$
$$zw = \left(4 + 7i\right)\left(-3 + i\right)$$
$$= 4\left(-3\right) + 4i + 7i\left(-3\right) + 7i^2$$
$$= -12 + 4i - 21i - 7$$
$$= -19 - 17i$$

Example 2

Find the standard form and the real and imaginary parts of $\left(2 + 3i\right)^2$.

Solution

$$\left(2 + 3i\right)^2 = 4 + 12i + 9i^2$$
$$= -5 + 12i$$

and hence the real part is -5 and the imaginary part is 12.

Example 3

Find the real and imaginary parts of $z + \frac{1}{z}$ if $z = \left(2 + i\right) / \left(1 - i\right)$.(Ans: $\text{Re} = \frac{7}{10}, \text{Im} = \frac{9}{10}$)

v. Equality of complex numbers: If *a*, *b*, *c* and *d* are real numbers, then $a + bi = c + di$ if and only if $a = c$ and $b = d$

Example 4

Find the solutions to the equation $z^2 = -4$.

Solution

Let $z = a + bi$ where $a, b \in \mathbb{R}$, so that

$$\begin{aligned} z^2 &= (a+bi)^2 \\ &= a^2 + 2abi - b^2 \\ &= (a^2 - b^2) + 2abi \end{aligned}$$

Now if $z^2 = -4$,

$$(a^2 - b^2) + 2abi = -4 + 0i$$

implying that

$$\begin{cases} a^2 - b^2 = -4 \ (1) \\ 2ab = 0 \ (2) \end{cases}$$

Equation (2) implies that either $a = 0$ or $b = 0$. If $a = 0$,the first equation gives us $-b^2 = -4$, and so $b = \pm 2$. If $b = 0$, the first equation gives us $a^2 = -4$, which has no solutions for real *a*.

Hence, the only solutions are $a = 0$ and $b = \pm 2$. Therefore, $z = 2i$ and $z = -2i$ are all the solutions to the equation $z^2 = -4$.

3 The Complex Plane

Since each complex number $z = a + bi$ is determined by two real numbers *a* and *b*, we can represent this complex number geometrically as the point in the plane with Cartesian coordinates (a, b). The plane in this representation is called **complex plane**. Each real number *a* is also the complex number $a + 0i$ that corresponds to a point $(a, 0)$ on the x-axis. Therefore the x-axis is called **the real axis**. A purely imaginary number $z = 0 + bi$ corresponds to a point on y-axis; therefore, the y-axis is called the **imaginary axis**.

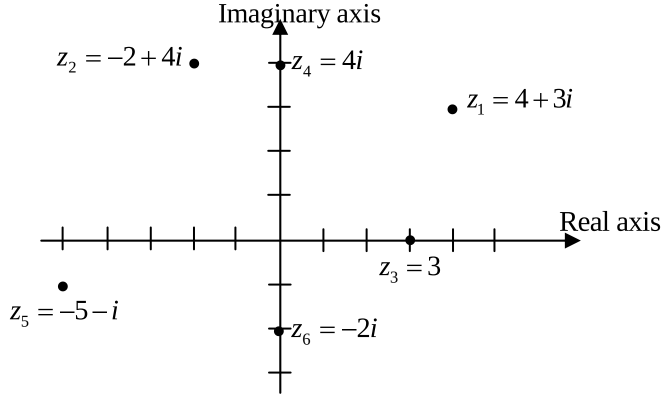
Example 1

Plot the following complex numbers in the complex plane:

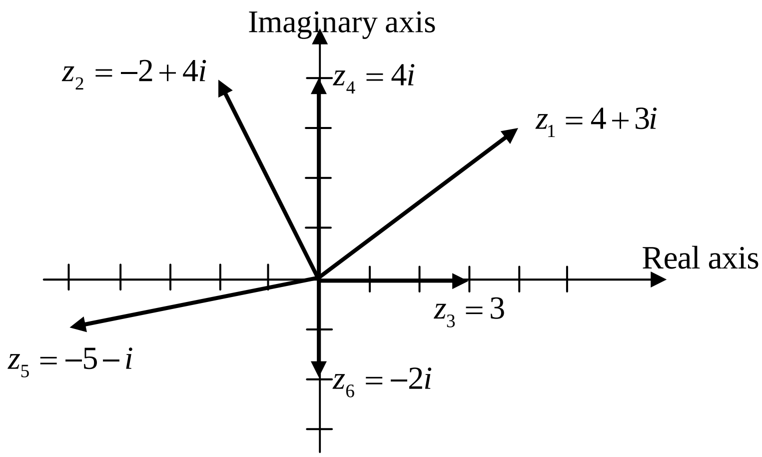
$$z_1 = 4 + 3i, z_2 = -2 + 4i, z_3 = 3, z_4 = 4i, z_5 = -5 - i, z_6 = -2i$$

Solution

These complex numbers are represented by the points $(4,3), (-2,4), (3,0), (0,4), (-5,-1)$, and $(0,-2)$.



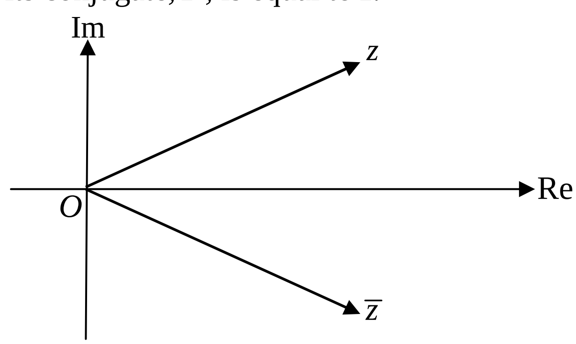
The complex number $z = a + bi$ can also be represented as a vector from the origin to the point coordinates (a, b). The complex numbers in the above example are represented as vectors in the figure below.



If $a + bi$ is any complex number, then $(a + bi)(a - bi) = a^2 + b^2$ which is always a real number. This relationship leads to the following definition.

The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$.

Geometrically, the complex conjugate $\overline{z}$ is the reflection of *z* in the real axis. If *z* is real, then *z* lies on the real axis and its conjugate, $\overline{z}$, is equal to *z*.



Example 2
If $z_1 = -2 + 3i$ and $z_2 = 0 - 5i$. Find $\overline{z}_1, \overline{z}_2, z_1\overline{z}_1$ and $z_2\overline{z}_2$.
Solution

We have $\overline{z}_1 = -2 - 3i$ and $\overline{z}_2 = 0 + 5i$ and therefore

$$z_1\overline{z}_1 = (-2 + 3i)(-2 - 3i) = 4 + 9 = 13$$

$$z_2\overline{z}_2 = (0 - 5i)(0 + 5i) = 25$$

Properties of The Complex Conjugate
If $z = a + bi$ and z_1 and z_2 are complex number, then

i. $z\overline{z} = a^2 + b^2$, which is always real and non-negative.

ii. $\overline{\overline{z}} = z$
iii. $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
iv. $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
v. $z = \overline{z}$ if and only if z is a real number

Since the number $z\overline{z} = a^2 + b^2$ is real and non-negative, it has a non-negative real square root $\sqrt{a^2 + b^2}$. This number is the distance from the origin to the point (a,b) and equals the length of the vector (a,b). It is called the **modulus or absolute value** of z and is denoted by $|z|$.

The **modulus or absolute value** of $a+bi$ is $|a+bi| = \sqrt{a^2+b^2}$

Properties of the Modulus

If z, z_1 and z_2 are complex numbers, then

i. $|z| = 0$ if and only if $z = 0$
ii. $|z| = |\overline{z}|$
iii. $z\overline{z} = |z|^2$
iv. $|z_1 z_2| = |z_1||z_2|$

We can now find the reciprocal of the non-zero complex number as follows. Start with

$$z\overline{z} = |z|^2$$

and divide by the non-zero real number $|z|^2$ to obtain

$$z\left(\frac{\overline{z}}{|z|^2}\right) = 1$$

Since $zz^{-1} = 1$, then $z^{-1} = \dfrac{\overline{z}}{|z|^2}$.

The **inverse** or **reciprocal** of the non-zero complex number $z = a+bi$ is

$$z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Example 3

If $z = 1+2i$, express z^{-1} in standard form

Solution

$$z^{-1} = \frac{1-2i}{5} = \frac{1}{5} - \frac{2}{5}i$$

4 Further Properties of Complex Numbers

Properties of Complex Numbers

If z_1, z_2 and z_3 are complex numbers, then

i. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (Associative Law of Addition)

ii. $z_1 + z_2 = z_2 + z_1$ (Commutative Law of Addition)

iii. There is a complex number 0such that, for all complex numbers z,
$z + 0 = z$

iv. For each complex number z, there is a negative $-z$ such that $z + (-z) = 0$

v. $z_1(z_2 z_3) = (z_1 z_2) z_3$ (Associative Law of Multiplication)

vi. $z_1 z_2 = z_2 z_1$ (Commutative Law of Multiplication)

vii. There is a unit 1 such that $z1 = z$ for all complex numbers z

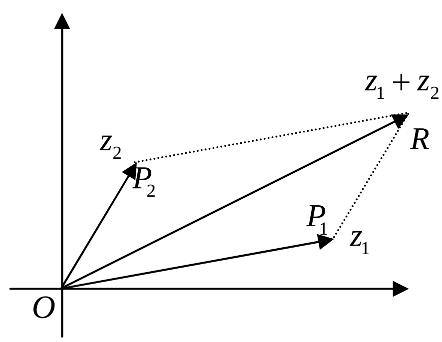
viii. Each non-zero complex number z has an inverse z^{-1} such that $zz^{-1} = 1$

ix. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (Distributive Law)

Triangle Inequality

For any complex numbers z_1, and z_2 $|z_1 + z_2| \le |z_1| + |z_2|$

Proof:



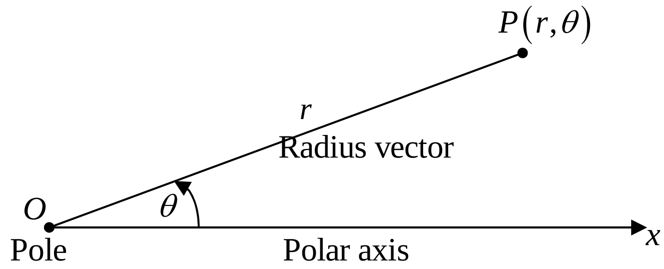
Let the vector $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ represent the complex numbers z_1 and z_2 respectively. Then $z_1 + z_2$ is represented by $\overrightarrow{OR}$, the diagonal from the origin in the parallelogram OP_1RP_2. Since sum of the two sides of the triangle OP_1R is greater than or equal to the third side

$$\left|\overrightarrow{OR}\right| \le \left|\overrightarrow{OP_1}\right| + \left|\overrightarrow{P_1R}\right|$$

Now $\left|\overrightarrow{P_1R}\right| = \left|\overrightarrow{OP_2}\right| = |z_2|$, so that $|z_1 + z_2| \le |z_1| + |z_2|$.

5 Polar Coordinates

A point in the plane can be located by using the familiar Rectangular Cartesian coordinate system and specifying its coordinates x and y. There is an alternative coordinate system that will be useful in dealing with multiplication and power of complex numbers. To use the **polar coordinate system** select a fixed point, *O*, in the plane, called the **pole** or origin and a fixed horizontal ray *Ox*, called the **polar axis**. The position of a point *P* in the plane is given by the ordered pair of real numbers (r, θ), called its **polar coordinates**, where $|r|$ is the distance from O to P and θ is the angle, in radians, that OP makes with the polar axis.



The vector OP is called the **radius vector**, θ is called the **vectorial angle**. By convention, the polar axis will always be selected in the direction of positive x-axis, and the angle θ will be positive if measured in a counterclockwise direction from Ox, and negative if measured clockwise. Normally r is taken to be non-negative.

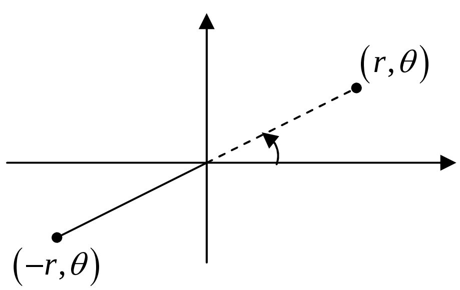
Example 1

Plot the points with these polar coordinates $\left(3,\frac{\pi}{6}\right),\left(2,\frac{2\pi}{3}\right),(5,0),\left(3,-\frac{\pi}{4}\right)$ and $\left(3,\frac{5\pi}{4}\right)$.

Example 2

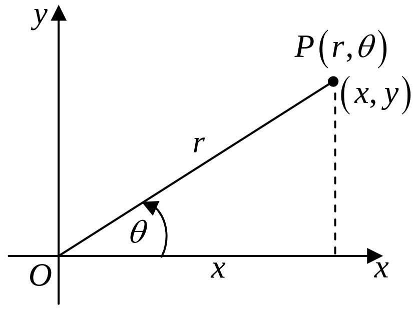
Plot, in separate diagrams, the points with the following polar coordinates
$\left(3,\frac{\pi}{3}\right),\left(3,\frac{7\pi}{3}\right)$, and $\left(3,\frac{-5\pi}{3}\right)$

Occasionally it is useful to allow r to be negative. In that case the point (r,θ) lies in the quadrant diagonally opposite to $(-r,\theta)$.



Given polar coordinates of a point, it is easy to calculate its Cartesian coordinates.
Let (r,θ) be polar coordinates of the point P, and let (x,y) be its Cartesian coordinates.

Then, in the right-angled triangle, $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$.



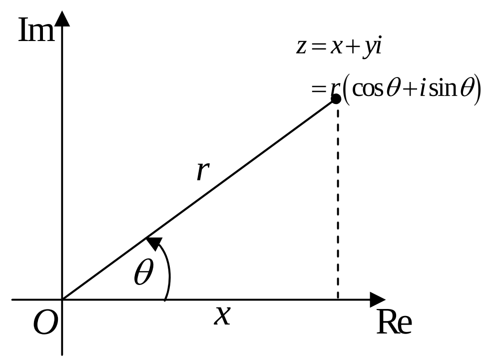
6 Complex Numbers in Other Forms

The complex number $z = x + yi$, written in standard form, can be represented by the point in the complex plane with Cartesian coordinates (x, y). Many problems in complex numbers, such as finding powers and roots, can be solved more easily by using the polar form of a complex number.

Let $z = x + yi$ be a complex number, and let (r, θ) where $r \geq 0$, be polar coordinates of the point representing *z*. Then $x = r\cos\theta$ and $y = r\sin\theta$ and

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + ri\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

We often abbreviate $\cos\theta + i\sin\theta$ by $cis\theta$



Polar form of a complex Number

$z = r(\cos\theta + i\sin\theta)$ or

$z = rcis\theta,$ where $r \geq 0$

The non-negative number $r = \sqrt{x^2 + y^2}$ is the **modulus** of the complex number z; that is, $r = |z|$. The angle θ, measured in radians, is called **argument** of z and is often abbreviated as $\arg z$. The angle θ is not determined uniquely, but is defined only up to a multiple of 2π. In general, $\arg z = \theta + 2k\pi$ for any integer *k*.

The **principle value** of the argument (sometimes called the **principle argument**) is the unique value of the argument that is in the range $-\pi < \arg z \leq \pi$ and is denoted by $\operatorname{Arg} z$.

Note that the inequalities at either ends of the range tell that a negative real number will have a principle value of argument of $\operatorname{Arg} z = \pi$.

Hence, given a complex number $z = x + yi$, then its modulus is found by

$$r = |z| = \sqrt{x^2 + y^2}$$

and its argument can be found by

$$\tan\theta = \frac{y}{x} \text{ or } \theta = \arctan\frac{y}{x}$$

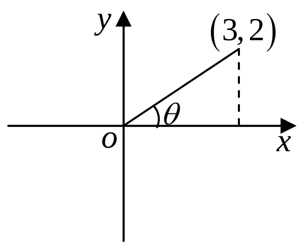
To help determine the correct argument, we should first plot the numbers in the complex plane.

Example 1

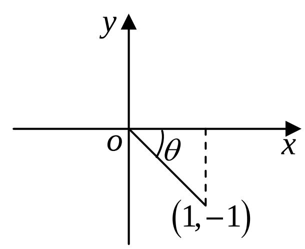
Determine the modulus and argument of each of the following complex numbers.

$$(a).3+i2 \quad (b).\ 1-i, \ (c).\ -1+i \quad (d).\ -\sqrt{6}-i\sqrt{2}$$

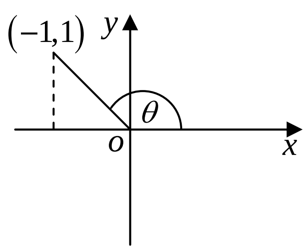
Solution



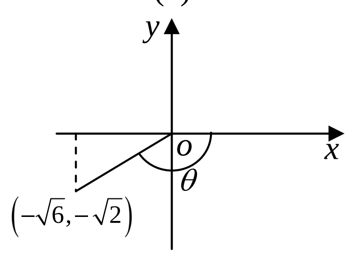
(a)



(b)



(c)



(d)

(a). $|3+i2| = \sqrt{3^2+2^2} = \sqrt{13}$

$$\text{Arg}(3+i2) = \arctan\left(\frac{2}{3}\right)$$

(b). $|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$$\text{Arg}(1-i) = -\tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4}$$

(c). $|-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

$$\text{Arg}(-1+i) = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

(d). $\left|-\sqrt{6}-i\sqrt{2}\right| = \sqrt{6+2} = \sqrt{8}$

$$\text{Arg}\left(-\sqrt{6}-i\sqrt{2}\right)=-\left(\pi-\tan^{-1}\frac{\sqrt{2}}{\sqrt{6}}\right)$$
$$=-\left(\pi-\tan^{-1}\sqrt{\frac{1}{3}}\right)$$
$$=-\left(\pi-\frac{\pi}{6}\right)=-\frac{5\pi}{6}$$

Complex Conjugate in Polar Form

If $z = rcis\theta$, then $\bar{z} = r\,cis\left(-\theta\right)$

Complex Multiplication in Polar form

If $z_1 = r_1\,cis\theta_1$ and $z_2 = r_2cis\theta_2$, then $z_1z_2 = r_1r_2cis\left(\theta_1+\theta_2\right)$

Proof is an exercise.

Complex Division in Polar form

If $z_1 = r_1\,cis\theta_1$ and $z_2 = r_2cis\theta_2$, and $r_2 \neq 0$, then $\frac{z_1}{z_2}=\frac{r_1}{r_2}cis\left(\theta_1-\theta_2\right)$

Proof is an exercise.

Example 2

If $z_1 = 2cis\frac{3\pi}{8}$ and $z_2 = 5cis\frac{2\pi}{3}$ calculate z_1z_2 and $\frac{z_1}{z_2}$. Make sure that all arguments are the principle ones.

Euler's Formula

$e^{i\theta} = \cos\theta + i\sin\theta$

Any complex number can be written as

$$z = r\left(\cos\theta + i\sin\theta\right) = rcis\theta = re^{i\theta}$$

The expression $re^{i\theta}$ is called **the exponential form** of the complex number *z*.

Note that $e^{2i\theta} = 1$ and $e^{i\theta} = -1$.

Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

i. $z_1z_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i\left(\theta_1+\theta_2\right)}$

ii. $\frac{z_1}{z_2}=\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}=\frac{r_1}{r_2}e^{i\left(\theta_1-\theta_2\right)}$

Example 3

Express $z = -2-2i$ in exponential form.

Solution

$$r = \left|z\right| = \sqrt{4+4} = 2\sqrt{2}$$

$$\theta = \text{Arg } z = -\frac{3\pi}{4}$$

Then, in polar form we have $z = 2\sqrt{2}e^{-i\frac{3\pi}{4}}$.

Example 4

If $z_1 = 2e^{\frac{2i\pi}{3}}$ and $z_2 = 5e^{\frac{i\pi}{4}}$. Find $z_1 z_2, \frac{z_1}{z_2},$ and z_2^{-1}

7 De Moivre's Theorem

De Moivre's Theorem

For any positive integer *n*, $(r\, cis\theta)^n = r^n\, cis\, n\theta$

Proof can be made by induction.

Example 1

Write $z = (1+i)^{20}$ in standard form.

Solution

In polar form, $1+i = \sqrt{2}cis\frac{\pi}{4}$; therefore,

$$z = \left(\sqrt{2}cis\frac{\pi}{4}\right)^{20} = \left(\sqrt{2}\right)^{20} cis\left(\frac{20\pi}{4}\right) = 2^{10} cis\pi = -1024$$

De Moivre's Theorem In Exponential Form

When the complex number is in exponential form, then the theorem is as follows

$\left(re^{i\theta}\right)^n = r^n e^{in\theta}$, for *n* any positive integer.

Example 2

If $z = 2e^{\frac{i\pi}{3}}$ find z^6 in standard form.

Solution

$$z^6 = 2^6 e^{\frac{6i\pi}{3}} = 64e^{2i\pi} = 64$$

De Moivre's Theorem for Negative Integers

$$(r\, cis\,\theta)^{-n} = r^{-n}\, cis(-n\theta)$$

Proof:

If *n* is a positive integer, then

$$(r\, cis\,\theta)^{-n} = \frac{1}{(r\, cis\,\theta)^n} = \frac{1}{r^n\,(cis\,\theta)^n} = r^{-n}\, cis(-n\theta)$$

Example 3

Express $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{-13}$ in standard form

Solution

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-13}=\left(1cis\frac{4\pi}{3}\right)^{-13}=1^{-13}\,cis\left(-\frac{52\pi}{3}\right)=cis\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

8 Roots of Complex Numbers

In this section we will use de Moivre's Theorem to find all the square roots, cubic roots, fourth roots, etc. of any real or complex number. That is we will find the nth roots of z, i.e $z^{\frac{1}{n}}$ where n is an non-zero integer.

Let $w=z^{1/n}$ then $z=w^n$

Using de Moivre's Theorem, we obtain

$$w^n=R^n\left(\cos n\phi+i\sin n\phi\right)\text{where } |w|=R \text{ and } \arg w=\phi$$

and since

$$z=r\left(\cos\theta+i\sin\theta\right)\text{with } |z|=r \text{ and } \arg z=\theta$$

then by equating $z=w^n$, we obtain

$$r\cos\theta=R^n\cos n\phi \text{ and } r\sin\theta=R^n\sin n\phi$$

and hence,

$$r=R^n\Rightarrow R=r^{\frac{1}{n}}$$

and $\begin{cases}\sin\theta=\sin n\phi\\ \cos\theta=\cos n\phi\end{cases}\Rightarrow n\phi=\theta+2\pi k,\ k\in\mathbb{Z}\ \text{or}\ \phi=\frac{\theta}{n}+\frac{2\pi k}{n},\ k\in\mathbb{Z}$

Therefore, if $z=rcis\theta$ then all its complex nth roots is

$$z^{1/n}=r^{1/n}\left[\cos\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)\right]\text{for } k=0,1,\ldots,n-1$$

or

$$z^{\frac{1}{n}}=\sqrt[n]{r}\,cis\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right),\text{for } k=0,1,\ldots,n-1$$

Example

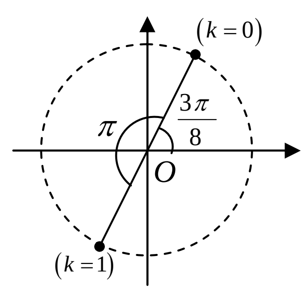
Find all square roots and cubic roots of $z=-\frac{1}{2}+i\frac{1}{2}$

Solution

z can be written as $z=2^{-1/2}cis\frac{3\pi}{4}$. Then the square roots are

$$z^{\frac{1}{2}}=\left(2^{-1/2}\right)^{\frac{1}{2}}cis\left(\frac{3\pi}{8}+\pi k\right),\text{for } k=0,1$$

$$=2^{-\frac{1}{4}}cis\left(\frac{3\pi}{8}+\pi k\right),\text{for } k=0,1$$



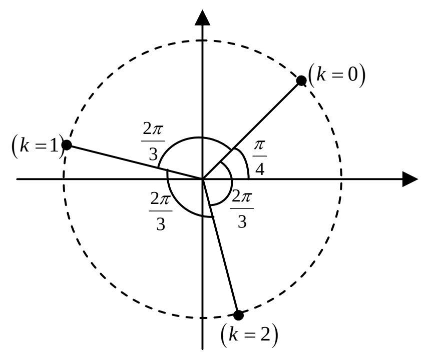
If $k=0$, then $z_0=2^{-\frac{1}{4}}cis\frac{3\pi}{8}$

If $k=1$, then $z_1 = 2^{-\frac{1}{4}} cis\left(\frac{3\pi}{8}+\pi\right) = 2^{-\frac{1}{4}} cis\frac{11\pi}{8} = 2^{-\frac{1}{4}} cis\left(-\frac{5\pi}{8}\right)$

The cubic roots are

$$z^{1/3} = r^{1/3} cis\left(\frac{\theta}{3}+\frac{2\pi k}{3}\right), k = 0,1,2$$

$$= 2^{-1/6} cis\left(\frac{\pi}{4}+\frac{2\pi k}{3}\right), k = 0,1,2$$



If $k=0$, then $z_0 = 2^{-1/6} cis\frac{\pi}{4}$

If $k=1$, then $z_1 = 2^{-1/6} cis\frac{11\pi}{12}$

If $k=1$, then $z_2 = 2^{-1/6} cis\frac{19\pi}{12} = 2^{-1/6} cis\left(\frac{19\pi}{12}-2\pi\right) = -2^{-1/6}\left(-\frac{5\pi}{12}\right)$

Example 2

Solve the equation $z^4 + 2\sqrt{2} - 2\sqrt{2}i = 0$.

Exercises

1. Find the real and imaginary parts of the complex numbers

a. $6+11i$ **b.** $-7i$ **c.** 14 **d.** $-1-2i$

2. Compute $z+w$ and zw.

a. $z=2+3i, w=1+i$ b. $z=2+i, w=2-i$ c. $z=4+5i, w=i$.

d. $z=-1+5i, w=7-11i$ e. $z=\sqrt{5}, w=2-6i$

f. $z=-1+3i, w=-4-2i$

3. Express the complex number in standard form

a. $(1+i)^2$ b. $(2+4i)+(7-i)$ c. $(-5i)^2$ d. $(-4+i)(-4-i)$

e. $(2-i)^2+(1+2i)^2$ f. $\left(\sqrt{7}-\sqrt{3}i\right)\left(\sqrt{7}+\sqrt{3}i\right)$

4. If $z=a+ib$, when is $(2-3i)z$ real?

5. If $z=a+bi$, when is $(-4+5i)z$ purely imaginary?

6. Plot the complex numbers $2+3i, 5-i, -3-4i,$ and $-3i$ in the complex plane as points.

7. State the complex conjugate and modulus of each complex number.

a. $2+9i$ **b.** $-i$ **c.** 3 **d.** $i(5+4i)$ **e.** $\sqrt{2}-3\sqrt{7}i$

8. If $z=3+5i$, plot $z, iz, i^2z, i^3z,$ and i^4z on the same diagram.

9. If $z = 1+\sqrt{3}i$, plot $z, z^2, 2iz,$ and $-iz$ on the same diagram.

10. Express in standard form

a. $\dfrac{2+i}{3-i}$ **b.** $\dfrac{1+2i}{1-2i}$ **c.** $\dfrac{4+3i}{7i}$ **d.** $\dfrac{(2+i)(1+2i)}{3-2i}$

11. If $z = 1-3i$, write in standard form.

a. z^{-1} **b.** $(\overline{z})^{-1}$ **c.** z^{-2} **d.** $(z\overline{z})^{-1}$

12. If $z_1 = 5+i, z_2 = 3-i,$ and $z_3 = -4+3i$, express the following in standard form

a. $(z_1z_2)z_3$ **b.** $z_1(z_2z_3)$ **c.** $z_1(z_2+z_3)$ **d.** $z_1z_2+z_1z_3$ **e.** $z_1+\overline{z}_1$

f. $\overline{z_2+z_3}$ **g.** $\overline{z}_2+\overline{z}_3$ **h.** $\overline{z}_1\overline{z}_2$ **i.** $\overline{z_1z_2}$

13. Solve for z

a. $(2-3i)z = 5$ **b.** $(1-2i)z+3 = -i$

13. a. Evaluate i^n for $n = 0,1,2,\ldots,10$

b. Evaluate $i^{4n}, i^{4n+1}, i^{4n+2}, i^{4n+3}, i^{4n+4}$ for all positive integer *n*.

14. If $z, z_1,$ and z_2 are complex numbers, prove each statement

a. $\overline{\overline{z}}$ **b.** $\overline{z_1+z_2} = \overline{z}_1+\overline{z}_2$ **c.** $|z| = |\overline{z}|$

d. $z = \overline{z}$ if and only if $z \in \mathbb{R}$ **e.** $|z| = 0$ if and only if $z = 0$ **f.** $z+\overline{z} = 2\,\mathrm{Re}(z)$

15. Prove that $\mathrm{Re}(z) = \dfrac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \dfrac{z-\overline{z}}{2i}$

16. Find all complex numbers z such that $z^2 = \overline{z}$.

17. Convert the complex numbers to polar form

a. $1+\sqrt{3}i$ b. $2i$ c. $-2\sqrt{2}-2\sqrt{2}i$ d. $\dfrac{2}{1+\sqrt{3}i}$ e. $\overline{\left(-\sqrt{3}-i\right)}$

18. Convert the complex numbers to standard form

a. $\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}$ b. $4\mathrm{cis}\dfrac{7\pi}{4}$ c. $3\mathrm{cis}\dfrac{5\pi}{6}$ d. $10(\cos 7\pi + i\sin 7\pi)$

e. $\dfrac{1}{2}\mathrm{cis}\left(-\dfrac{\pi}{3}\right)$

19. Write each of the complex numbers in exercise 17 in exponential form.

20. Express each quantity as a complex number in polar form

a. $(1-i)(1+\sqrt{3}i)$ b. $\left(2e^{\frac{i\pi}{3}}\right)\left(5e^{\frac{2\pi i}{3}}\right)$ c. $\dfrac{-\sqrt{2}-\sqrt{2}i}{\sqrt{3}-i}$ d. $\left(-6+6\sqrt{3}i\right)^2$

e. $\left[4\mathrm{cis}\left(-\dfrac{\pi}{6}\right)\right] \div \left(\dfrac{1}{2}\mathrm{cis}\dfrac{\pi}{2}\right)$

21. If $z = r\mathrm{cis}\theta \neq 0$, prove that $\dfrac{\overline{z}}{z} = \cos^2\theta - \sin^2\theta - 2i\sin\theta\cos\theta$.

22. If θ is not an odd multiple of π, prove that

$$\frac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta} = \cos\theta+i\sin\theta$$

23. Use Euler's formula to prove that

a. $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ b. $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

24. Use de Moivre's Theorem to express each complex number in standard form

a. $\left(2\text{cis}\dfrac{\pi}{4}\right)^3$ b. $\left(\cos\dfrac{\pi}{12}+i\sin\dfrac{\pi}{12}\right)^8$ c. $(1-i)^{16}$

d. $\left(\dfrac{3\sqrt{3}}{2}+\dfrac{3i}{2}\right)^3$ e. $\left(e^{-\frac{2\pi i}{3}}\right)^7$ f. $\left(\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}i\right)^9$

g. $\dfrac{1}{\left(\sqrt{3}-i\right)^{10}}$ h. $\dfrac{(-1+i)^6(-1-i)^4}{\left(\sqrt{2}+\sqrt{2}i\right)^3}$ i. $(-1+i)^{-4}$

25. If $z=\text{cis}\dfrac{2\pi}{9}$, plot $z, z^2, z^3, ..., z^9$ in the complex plane.

26. Using de Moivres's Theorem, prove each statement.

a. $\cos 2\theta = 2\cos^2\theta - 1$ $\sin 2\theta = 2\sin\theta\cos\theta$

b. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1,\ \sin 4\theta = 4\sin\theta\cos\theta\left(1-2\sin^2\theta\right)$

27. Find the value in standard form.

a. $(1+i)(-1+i)^2(-1-i)^3(1-i)^4$

b. $(1+i)+(-1+i)^2+(-1-i)^3+(1-i)^4$

28. Find all the roots in polar form and illustrate each geometrically.

a. the cube roots of 8 b. the fourth roots of i

c. the squre roots of $-i$ d. the fifth roots of $32\text{cis}\dfrac{5\pi}{6}$

e. the sixth roots of 1 f. the fifth roots -32

29. Solve the equations and express your answers in standard form

a. $z^4-16=0$ b. $z^3=64i$ c. $z^4+1=0$ d. $z^4+8-8\sqrt{3}i=0$

e. $z^2-(2+2i)z+i=0$ f. $z^2+2z-\sqrt{3}i=0$

30. Use the complex exponential to solve the equations; leave your answers in exponential form.

a. $z^4+\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}=0$ b. $z^5-243=0$

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